85. *Other Characterizations and Weak Sum Theorems for Metric-dependent Dimension Functions*

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1. Introduction. In [7] and [8] the author introduced the metric-dependent dimension functions d_6 and d_7 and characterized them in terms of Lebesgue covers of metric spaces and for uniform spaces. The results for metric spaces are the following.

Theorem 1.1. *Let (X, ρ) be a metric space. Then $d_6(X, \rho) \leq n$ if and only if every countable Lebesgue cover has an open refinement of order $\leq n+1$.*

Theorem 1.2. *Let (X, ρ) be a metric space. Then $d_7(X, \rho) \leq n$ if and only if every locally finite Lebesgue cover of X has an open refinement of order $\leq n+1$.*

A natural question now arises as to whether new metric-dependent dimension functions occur if "countable" and "locally finite" in the above characterization theorems are replaced by "star-countable" and "point finite" respectively. In § 2 we define two such new dimension functions, d_6^* and d_7^*, and prove that $d_6^* = d_6$ and $d_7^* = d_7$. We also show that the dimension function d_5 of Hodel [1] has a "star-countable" equivalent definition. In § 3 we introduce a new metric-dependent dimension function d_3^*, characterize it in terms of Lebesgue covers, and observe the following inequality $d_3 \leq d_3^* \leq d_6$. In § 4 we generalize a sum theorem of Morita and establish "weak" locally finite sum theorems for $d_2, d_3, d_3^*, d_6, d_7$ and d_0 in both metric and uniform spaces.

2. Equivalent characterization for d_6 and d_7.

Definition 2.1. Let (X, ρ) be a metric space. Then $d_6^*(X, \rho) \leq n$ if and only if every star-countable Lebesgue cover of X has an open refinement of order $\leq n+1$.

We note that $d_6(X, \rho) \leq d_6^*(X, \rho)$ by Definition 2.1 and Theorem 1.1. By a similar technique as in Theorem 2 of [2] by Morita we have the following.

Theorem 2.2. *Let $\mathcal{G} = \{G_\alpha : \alpha \in A\}$ be a star-countable open cover of a T_1 space X. We divide the index set A into subsets $\{A_\beta : \beta \in B\}$ such that α and γ belong to A_β if and only if there exists a positive integer n such that $G_\alpha \subset \mathrm{St}^n(G_\gamma, \mathcal{G})$. Define $X_\beta = \bigcup_{\alpha \in A_\beta} G_\alpha$. Then we have the following*

(1) $X = \bigcup_{\beta \in B} X_\beta$

(2) $X_\beta \cap X_{\beta'} = \emptyset$ *for* $\beta \neq \beta'$

(3) X_β *is open and closed in* X *for each* $\beta \in B$.

(4) $\mathcal{G}_\beta = \{G_\alpha : \alpha \in A_\beta\}$ *is a countable open cover of* X_β *for each* $\beta \in B$.

Theorem 2.3. *Let* (X, ρ) *be a metric space. Then* $d_6(X, \rho) = d_6^*(X, \rho)$.

Proof. Assume $d_6(X, \rho) \leq n$. Let $\mathcal{G} = \{G_\alpha : \alpha \in A\}$ be a star-countable Lebesgue cover of (X, ρ). By Theorem 2.2 above the index set A can be partitioned into subsets $\{A_\beta : \beta \in B\}$, satisfying the conditions (1)–(4), where each $\mathcal{G}_\beta$ is a countable Lebesgue cover of X_β.

Since $d_6(X, \rho) \leq n$, $d_6(X_\beta, \rho) \leq n$ for each $\beta \in B$; so that $\mathcal{G}_\beta$ has an open refinement $\mathcal{U}_\beta$ such that order $(\mathcal{U}_\beta) \leq n+1$ for each $\beta \in B$. Therefore $\mathcal{U} = \bigcup_{\beta \in B} \mathcal{U}_\beta$ is an open refinement of $\mathcal{G}$ and order $(\mathcal{U}) \leq n+1$. Hence $d_6^*(X, \rho) \leq n$.

Corollary. *Let* (X, ρ) *be a metric space. Then* $d_6(X, \rho) \leq n$ *if and only if every star-countable Lebesgue cover of* X *has an open refinement of order* $\leq n+1$.

By a similar proof as in [8] we obtain the following.

Theorem 2.4. *Let* $(X, \mathcal{U})$ *be a normal uniform space. Then* $d_6(X, \mathcal{U}) \leq n$ *if and only if every star-countable Lebesgue cover of* X *has an open refinement of order* $\leq n+1$.

We now consider the metric-dependent dimension function similar to d_7, which is defined in [7].

Definition 2.5. The dimension function d_7^* is defined like d_7 in [7] with the exception that $\{X - C'_\alpha : \alpha \in A\}$ is point finite.

Definition 2.6. Let X be a set and $\mathcal{G} = \{\mathcal{G}_\lambda : \lambda \in \Lambda\}$ be a collection of families of subsets of X. For each $\lambda \in \Lambda$, let $\mathcal{G}_\lambda = \{G_\alpha : \alpha \in A_\lambda\}$. Then

$$\bigwedge_{\lambda \in \Lambda} \{\mathcal{G}_\lambda\} = \{\cap G_{\alpha(\lambda)} : \alpha(\lambda) \in A_\lambda, \lambda \in \Lambda\}$$

Lemma. *Let* X *be a normal space,* $\{G_\alpha : \alpha \in A\}$ *a point finite open collection, and* $\{F_\alpha : \alpha \in A\}$ *a closed collection such that* $F_\alpha \subset G_\alpha$ *for each* $\alpha \in A$. *If* $\mathcal{G} = \bigwedge_{\alpha \in A} \{G_\alpha, X - F_\alpha\}$ *has an open refinement of order* $\leq n+1$, *then there exist closed sets* B_α, *separating* F_α *and* $X - G_\alpha$ *for each* $\alpha \in A$ *such that order* $\{B_\alpha : \alpha \in A\} \leq n$.

Proof. Since $\{G_\alpha : \alpha \in A\}$ is point finite it is clear that $\mathcal{G} = \bigwedge_{\alpha \in A} \{G_\alpha, X - F_\alpha\}$ is a point finite cover of X. If $\mathcal{V} = \{V_\delta : \delta \in \Delta\}$ is an open refinement of $\mathcal{G}$ of order $\leq n+1$ we may assume that $\mathcal{V}$ is also point finite. Note that given $V \in \mathcal{V}$, then V intersects at most a finite number of the F_α. For if $V \cap F_\alpha \neq \emptyset$, then $V \subseteq G_\alpha$ and $\{G_\alpha : \alpha \in A\}$ is point finite. Since $\mathcal{V}$ is point finite and X is normal, there exists a closed cover $\mathcal{D} = \{D_\delta : \delta \in \Delta\}$ such that $D_\delta \subset V_\delta$ for each $\delta \in \Delta$. The remainder of the proof is essentially the same as [6, II, 5, B].

Theorem 2.7. *Let* (X, ρ) *be a metric space. Then* $d_7^*(X, \rho) \leq n$

if and only if every point finite Lebesgue cover of X has an open refinement of order $\leq n+1$.

Proof. Using the previous lemma, the proof proceeds as that of Theorem 4.2 in [7].

In [9] the author has shown the following.

Theorem 2.8. *Let* (X, ρ) *be a metric space. If* $\mathcal{G}$ *is a point finite Lebesgue cover of X, then* $\mathcal{G}$ *has a locally finite Lebesgue refinement.*

Hence the following is clear.

Corollary. *Let* (X, ρ) *be a metric space. Then* $d_7(X, \rho)=d_7^*(X, \rho)$.

As was the case for d_6 above we now have:

Theorem 2.9. *Let* $(X, \mathcal{U})$ *be a normal uniform space. Then* $d_7(X, \mathcal{U})\leq n$ *if and only if every point finite Lebesgue cover of X has an open refinement of order* $\leq n+1$.

In [1] Hodel introduced the metric dependent dimension function d_5. We now observe that d_5 has an alternate definition.

Definition 2.10. Let (X, ρ) be a metric space. if $X=\emptyset$, $d_5^*(X, \rho) =-1$. Otherwise, $d_5^*(X, \rho)\leq n$ if (X, ρ) satisfies this condition:

(D_5^*) Given any collection of closed pairs $\{C_\alpha, C'_\alpha : \alpha \in A\}$ such that there exists $\delta>0$ with

(1) $\rho(C_\alpha, C'_\alpha)>0$ for each $\alpha \in A$,

(2) $\{X-C'_\alpha : \alpha \in A\}$ is star countable,

then there exist closed sets B_α, separating C_α and C'_α, such that order $\{B_\alpha : \alpha \in A\}\leq n$.

Note that $d_5(X, \rho)\leq d_5^*(X, \rho)$ by definition.

Theorem 2.11. *Let* (X, ρ) *be a metric space. Then* $d_5(X, \rho) =d_5^*(X, \rho)$.

Proof. Assume $d_5(X, \rho)\leq n$ and $\{C_\alpha, C'_\alpha : \alpha \in A\}$ is any collection of closed pairs satisfying (D_5^*) above. Since $\{X-C'_\alpha : \alpha \in A\}$ is star-countable, $\{X-C'_\alpha : \alpha \in A\}\cup\{X-C_{\alpha_0}\}$ is a star-countable open cover of X for any fixed $\alpha_0 \in A$. By Theorem 2.2 above we can partition A into subsets $\{A_\beta : \beta \in B\}$ satisfying the conditions (1)–(4).

Now $d_5(X, \rho)\leq n$ implies that for each $\beta \in B$ exist closed sets $B_{\beta(\alpha)}$, separating C_α and C'_α, for all $\alpha \in A_\beta$ such that order $\{B_{\beta(\alpha)} : \alpha \in A_\beta\}\leq n$. Hence $\{B_{\beta(\alpha)} : \alpha \in A_\beta, \beta \in B\}$ is a collection of closed sets satisfying (D_5^*) above, so that $d_5^*(X, \rho)\leq n$.

3. The dimension function d_3^*.

Definition 3.1. Let (X, ρ) be a metric space. If $X=\emptyset$, then $d_3^*(X, \rho)=-1$. Otherwise, $d_3^*(X, \rho)\leq n$ if (X, ρ) satisfies this condition:

(D_3^*) Given any collection of closed pairs $\{C_\alpha, C'_\alpha : \alpha \in A\}$ such that there exists $\delta>0$ with

(1) $\rho(C_\alpha, C'_\alpha)>\delta$ for each $\alpha \in A$,

(2) $\{X-C'_\alpha : \alpha \in A\}$ is star-finite,

then there exist closed sets B_α, separating C_α and C'_α, such that order $\{B_\alpha : \alpha \in A\} \leq n$.

Theorem 3.2. *Let (X, ρ) be a metric space. Then $d_3^*(X, \rho) \leq n$ if and only if every star-finite Lebesgue cover of X has an open refinement of order $\leq n+1$.*

Proof. Since $\{X - C'_\alpha : \alpha \in A\}$ is star-finite, then $\bigwedge_{\alpha \in A} \{X - C_\alpha, X - C'_\alpha\}$ is a star-finite Lebesgue cover of X. Hence the proof proceeds exactly as that of Theorem 4.2 in [7].

Corollary. *Let (X, ρ) be a metric space. Then $d_3(X, \rho) \leq d_3^*(X, \rho) \leq d_6(X, \rho)$.*

4. Weak sum theorems.

Definition 4.1. Let X be a topological space and $\mathcal{G}$ be an open cover of X. We say that the $\mathcal{G}$-dimension, denoted $\mathcal{G}$-dim, of X is the smallest integer n such that $\mathcal{G}$ has an open refinement of order $\leq n+1$. If no such integer exists, we say $\mathcal{G}$-dim (X) is infinite; and $\mathcal{G}$-dim $(\emptyset) = -1$.

K. Morita* [5] has shown the following:

Theorem 4.2. *Let X be a normal space, $\{U_\alpha : \alpha \in A\}$ a locally finite open collection, and $\{F_\alpha : \alpha \in A\}$ a closed collection such that $F_\alpha \subset U_\alpha$ for each $\alpha \in A$. Let $\mathcal{G}$ be any locally finite open cover of X such that $\mathcal{G}$-dim$(F_\alpha) \leq n$ for each $\alpha \in A$. If $\dim(F_\alpha \cap F_\beta) \leq n-1$ for $\alpha \neq \beta$, then $\mathcal{G}$-dim $(\bigcup_{\alpha \in A} F_\alpha) \leq n$.*

We generalize this to the following:

Theorem 4.3. *Let X be a normal space, $\{U_\alpha : \alpha \in A\}$ a locally finite open collection, and $\{F_\alpha : \alpha \in A\}$ a closed collection such that $F_\alpha \subset U_\alpha$ for each $\alpha \in A$. Let $\mathcal{G}$ be any locally finite open cover of X such that $\mathcal{G}$-dim $(F_\alpha) \leq n$ for each $\alpha \in A$. If $\dim[bdry(F_\alpha) \cap F_\beta] \leq n-1$ for $\alpha \neq \beta$, then $\mathcal{G}$-dim $(\bigcup_{\alpha \in A} F_\alpha) \leq n$.*

Proof. Define for each positive integer k, A_k, to be the collection of all distinct subsets $\{\alpha_1, \alpha_2, \cdots, \alpha_k\}$ of A with cardinality k such that $\bigcap_{i=1}^{k} F_{\alpha_i} \neq \emptyset$. Define

$$\mathcal{H}_k = \{\bigcap_{i=1}^{k} F_{\alpha_i} - \bigcup_{\beta \neq \alpha_i} \operatorname{int}(F_\beta) : \{\alpha_1, \cdots, \alpha_k\} \in A_k, \beta \in A\},$$

and $\mathcal{H} = \bigcup_{k=1}^{\infty} {}_k = \{H_\lambda, \lambda \in \Lambda\}$. Clearly $H_\lambda \in \mathcal{H}$ implies H_λ is closed in X and there exists some F_α such that $H_\lambda \subset F_\alpha$.

Assertion 1. $\bigcup_{\lambda \in \Lambda} H_\lambda = \bigcup_{\alpha \in A} F_\alpha$.

Let $x \in \bigcup_{\alpha \in A} F_\alpha$. Since $\{F_\alpha : \alpha \in A\}$ is locally finite there exists some

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integer $m>0$ such that the order of x with respect to $\{F_\alpha : \alpha \in A\}$ is equal to m. Hence by definition x belongs to some member of $\mathcal{H}_m$. Also $\{F_\alpha : \alpha \in A\}$ locally finite implies that $\mathcal{H}$ is locally finite.

Assertion 2. Let H_λ and H_μ belong to $\mathcal{H}$ such that $H_\lambda \neq H_\mu$. Then there exist distinct members F_α and F_β such that $H_\lambda \cap H_\mu \subset (\text{bdry } F_\alpha) \cap F_\beta$. This assertion is obvious if $H_\lambda \cap H_\mu = \emptyset$. Let $H_\lambda = \bigcap_{i=1}^{n} F_{\alpha_i} - \bigcup_{\beta \neq \alpha_i} \text{int}(F_\beta)$ and $H_\mu = \bigcap_{i=1}^{m} F_{\gamma_i} - \bigcup_{\beta \neq \gamma_i} \text{int}(F_\beta)$. Since $H_\lambda \neq H_\mu$ we have $\{\alpha_1, \cdots, \alpha_n\} \neq \{\gamma_1, \cdots, \gamma_m\}$; so that either $\alpha_i \notin \{\gamma_1, \cdots, \gamma_m\}$ for some $i \in \{1, 2, \cdots, n\}$ or $\gamma_j \notin \{\alpha_1, \cdots, \alpha_n\}$ for some $j \in \{1, 2, \cdots, m\}$. Accordingly in either case we have $H_\lambda \cap H_\mu \subset (\text{bdry } F_{\alpha_i}) \cap F_{\gamma_1}$ or $H_\lambda \cap H_\mu \subset (\text{bdry } F_{\gamma_j}) \cap F_{\alpha_1}$.

Now by Assertion 2 we have $\dim (H_\lambda \cap H_\mu) \leq \dim [(\text{bdry } F_\alpha) \cap F_\beta] \leq n-1$. Since $\{U_\alpha : \alpha \in A\}$ is locally finite, we have that $\{\bigcap_{i=1}^{n} U_{\alpha_i} : \{\alpha_1, \cdots, \alpha_n\} \in A_n, n=1, 2, \cdots\}$ is locally finite collection of open subsets of X. Since $H_\lambda = \bigcap_{i=1}^{n} F_{\alpha_i} - \bigcup_{\beta \neq \alpha_i} \text{int}(F_\beta) \subset \bigcap_{i=1}^{n} U_{\alpha_i}$ we have by Theorem 4.2 above $\mathcal{G}\text{-dim}(\bigcup_{\alpha \in A} F_\alpha) = \mathcal{G}\text{-dim}(\bigcup_{\lambda \in \Lambda} H_\lambda) \leq n$.

Theorem 4.4. *Let (X, ρ) be a metric space satisfying these conditions.*

(1) $X = \bigcup_{\alpha \in A} F_\alpha$, *where F_α is closed in X.*

(2) $\{F_\alpha : \alpha \in A\}$ *is locally finite.*

(3) $d_0(F_\alpha, \rho) \leq n$ *for all α in A.*

(4) $\dim [(\text{bdry } F_\alpha) \cap F_\beta] \leq n-1$ *for $\alpha \neq \beta$.*

Then $d_0(X, \rho) \leq n$.

Proof. Let $\varepsilon > 0$ be given. We want to find an open cover $\mathcal{U}$ of X such that ρ-mesh $(\mathcal{U}) < \varepsilon$ and $\text{ord}(\mathcal{U}) \leq n+1$. Since $d_0(F_\alpha, \rho) \leq n$ for each α in A, there exists an open cover $\mathcal{U}_\alpha$ of F_α such that ρ-mesh $(\mathcal{U}_\alpha) < \varepsilon/2$ and $\text{ord}(\mathcal{U}_\alpha) \leq n+1$. As before we can assume $\mathcal{U}_\alpha$ is locally finite and hence we can shrink $\mathcal{U}_\alpha$ to a closed cover of F_α which will then be a closed locally finite collection in X. Again since X is paracompact we may assume that $\mathcal{U}_\alpha$ is a locally finite open collection in X such that ρ-mesh $\mathcal{U}_\alpha < \varepsilon$ and $\text{ord}(\mathcal{U}_\alpha) \leq n+1$. Define $\mathcal{U} = \bigcup_{\alpha \in A} \mathcal{U}_\alpha$. Clearly $\mathcal{U}$ is an open cover of X. Furthermore $\mathcal{U}$ can be assumed to be locally finite since $\{F_\alpha : \alpha \in A\}$ can be expanded to a locally finite collection $\{G_\alpha : \alpha \in A\}$, and we can restrict the collection $\mathcal{U}_\alpha$ to G_α for each $\alpha \in A$. By Theorem 4.3, $\mathcal{U}$ has an open refinement $\mathcal{V}$ of order $\leq n+1$. Also ρ-mesh $(\mathcal{V}) \leq \rho$-mesh $(\mathcal{U}) < \varepsilon$, so that $d_0(X, \rho) \leq n$.

Using the Lebesgue covering characterizations for each of the dimension functions d_2, d_3, d_3^*, d_6, and d_7 it follows that Theorem 4.4

holds for these dimension functions in metric space as well as for normal uniform spaces. See [7] and [8].

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