80. *Notes on Modules.* I

By Ferenc A. SZÁSZ

(Comm. by Kinjirô KUNUGI, M. J. A., April 13, 1970)

Sharpening a result of Kertész [3], who showed the semisimplicity (in the sense of Jacobson) of the total endomorphism ring $E(M)$ of a completely reducible module M over an arbitrary associative ring A, we prove in our paper the Neumann regularity of this ring $E(M)$. The result also generalizes a theorem of Johnson-Kiokemeister [2], and is an English version of our earlier result [4], written in Hungarian.

Theorem. *Assume that M is a completely reducible module over an arbitrary ring A, and $E(M)$ is the total ring of endomorphisms of M. Then $E(M)$ is regular in the sense of Neumann.*

Proof. Let M be homogeneous. Supposing that the elements of A are right operators, and the elements of $E(M)$ left operators for the module M, for any fixed element $\gamma \in E(M)$ there exists an A-submodule K of M satisfying:

(1) $$M=\gamma M\oplus K$$

being M completely reducible. Denote L_γ the kernel of the endomorphism γ in M, that is

$$L_\gamma=\{m;\, m\in M,\ \gamma m=0\}$$

Then L_γ is an A-submodule of M, and there exists another A-submodule N of M with

(2) $$M=L_\gamma\oplus N$$

Being also N completely reducible, we have

$$N=\sum\oplus\{n_\alpha\} \qquad \{\alpha\in\Gamma)$$

with simple A-modules $\{n_\alpha\}$. By (2) our module can be generated by the set of all elements γn_α $(\alpha\in\Gamma)$.

Assume that we have a linear connection

(3) $$\gamma n_{\alpha_1}a_1+\cdots+\gamma n_{\alpha_k}a_k=0 \qquad (a_i\in A)$$

then for the element

$$n^*=n_{\alpha_1}a_1+\cdots+n_{\alpha_k}a_k$$

obviously $\gamma n^*=0$ and $n^*\in L_\gamma$ holds, which yields by (2) also $n^*=0$. The direct sum $\sum\oplus\{n_\alpha\}$ can be built, therefore $n^*=0$ implies $n_{\alpha_1}a_1=\cdots=n_{\alpha_k}a_k=0$ and thus also $\gamma n_{\alpha_1}a_1=\cdots=\gamma n_{\alpha_k}a_k=0$. Consequently, the set of all γn_α is a basis of γM. Furthermore, let the set of all $k_\beta(\beta\in\Gamma')$ be a basis for k, then by (1) one has

$$M=\sum_{\alpha\in\Gamma}\oplus\{\gamma n_\alpha\}\oplus\sum_{\beta\in\Gamma'}\oplus\{k_\beta\}$$

Evidently, every element of $E(M)$ can be determined by his effect on the basis elements γn_α and K_β of M ($\alpha \in \Gamma$, $\beta \in \Gamma'$). Because M is now by assumption homogeneous, there exists an element $\delta \in E(M)$ with

(4) $\delta(\gamma n_\alpha) = n_\alpha, \quad \delta k_\beta = 0 (\alpha \in \Gamma, \beta \in \Gamma')$

Define $\vartheta = \gamma\delta\gamma - \gamma$. Then by (4) one has $\vartheta n_\alpha = 0$ and $\vartheta N = 0$, furthermore by $\gamma L_\gamma = 0$ and (2) also $\vartheta M = 0$. Hence $\vartheta = 0$ and $\gamma = \gamma\delta\gamma$ which means the regularity Neumann for $E(M)$ in the homogeneous case.

If M is not homogeneous, then M is a discrete direct sum of its homogeneous components H_ε, and $E(M)$ is the complete direct sum of the rings $E(H_\varepsilon)$. But any $E(H_\varepsilon)$ is regular by the above, and thus also their complete direct sum is regular in the sense of Neumann.

This completes the proof of Theorem.

References

[1] N. Jacobson: Structure of Rings. Providence (1964).

[2] R. E. Johnson and F. Kiokemeister: The endomorphism of the total operator domain of an infinite module. Trans. Amer. Math. Soc., **62**, 404–430 (1947).

[3] A. Kertész: Beiträge zur Theorie der Operatormoduln. Acta Math. Acad. Sci. Hungar., **8**, 235–257 (1957).

[4] F. Szász: A teljesen reducibilis operátormodulusokról (On the completely reducible operator modules). Magyar Tudományos Akadémia. III. Oszt. Közl., **11**(4), 417–425 (1961).