77. On Nest Algebras of Operators

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1. In this paper we study certain algebras of operators termed ‘nest algebras’, which were introduced by J. R. Ringrose [3]. Our main results (Theorems 4 and 5) are concerned with characterizations of such algebras, and consequently it is proved that each weakly closed maximal triangular operator algebra is hyperreducible.

Throughout this paper the terms Hilbert space, subspace, operator, projection are used to mean complex Hilbert space, closed linear subspace, bounded linear operator, orthogonal projection, respectively. Given a subspace \( \mathcal{M} \) of a Hilbert space \( \mathcal{H} \), we shall write \( P_\mathcal{M} \) for the projection from \( \mathcal{H} \) onto \( \mathcal{M} \), and \( \mathcal{H} \ominus \mathcal{M} \) for the orthogonal complement of \( \mathcal{M} \) in \( \mathcal{H} \). If \( \{ \mathcal{M}_n \} \) is a collection of subspaces of \( \mathcal{H} \), then the smallest subspace which contains each \( \mathcal{M}_n \) will be denoted by \( \bigvee \mathcal{M}_n \), and the largest subspace contained in each \( \mathcal{M}_n \) will be denoted by \( \bigwedge \mathcal{M}_n \). Set inclusion in the wide sense will be denoted by the symbol ‘\( \subseteq \)’, and we reserve ‘\( \subset \)’ for proper inclusion.

The class of all operators from a Hilbert space \( \mathcal{H} \) into itself will be denoted by \( \mathcal{L}(\mathcal{H}) \). By an algebra of operators on \( \mathcal{H} \) we shall mean a subset of \( \mathcal{L}(\mathcal{H}) \) such that, if \( \lambda \) is a complex number and \( A, B \in \mathcal{A} \), then \( \lambda A, AB, A + B \in \mathcal{A} \). A self-adjoint algebra of operators will be termed a \( \mathcal{B} \)-algebra.

2. Following J. R. Ringrose, a family \( \mathcal{N} \) of subspaces of a Hilbert space \( \mathcal{H} \) will be called a nest if it is totally ordered by the inclusion relation \( \subseteq \); \( \mathcal{N} \) will be called a complete nest if, further,

(i) \( (0), \mathcal{H} \in \mathcal{N} \);

(ii) given any subnest \( \mathcal{N}_0 \) of \( \mathcal{N} \), the subspaces \( \bigwedge \mathcal{M} \), \( \bigvee \mathcal{M} \) are both members of \( \mathcal{N} \).

Given a complete nest \( \mathcal{N} \) and a non-zero subspace \( \mathcal{M} \) in \( \mathcal{N} \), we define

\[ \mathcal{M}_+ = \bigvee \{ \mathcal{N} \mid \mathcal{N} \in \mathcal{N}, \mathcal{M} \subset \mathcal{N} \} \]

Clearly \( \mathcal{M}_+ \in \mathcal{N} \).

If \( \mathcal{N} \) is a complete nest of subspaces of a Hilbert space \( \mathcal{H} \), then the nest algebra \( \mathcal{A}_\mathcal{N} \) associated with \( \mathcal{N} \) is defined to be the class of all operators on \( \mathcal{H} \) which leave invariant each subspace in \( \mathcal{N} \). Clearly \( \mathcal{A}_\mathcal{N} \) is a weakly closed subalgebra of \( \mathcal{L}(\mathcal{H}) \).

The following lemma, included here for the sake of completeness,
is restatement of ([3], Theorem 3.4).

**Lemma 1.** Let $\mathcal{H}$ be a complete nest of subspaces of a Hilbert space $\mathcal{F}$, and let $\mathcal{M}$ be a subspace of $\mathcal{F}$ which is invariant under each operator in $\mathcal{A}_\mathcal{M}$. Then $\mathcal{M} \in \mathcal{H}$.

Let $\mathcal{R}$ be a subalgebra of $\mathcal{L}(\mathcal{F})$, and let $\mathcal{R}^* = \{ A^* | A \in \mathcal{R} \}$. Then after terminologies for the triangular algebras of operators in [1], the sub $*$-algebra $\mathcal{A} = \mathcal{R} \cap \mathcal{R}^*$ will be called the diagonal of $\mathcal{R}$, and the core of $\mathcal{R}$ is defined to be the von Neumann algebra generated by the projections onto subspaces of $\mathcal{F}$ which are invariant under $\mathcal{R}$.

**Lemma 2.** Let $\mathfrak{A}$ be a von Neumann algebra on a Hilbert space $\mathcal{F}$, and let $\mathcal{R}$ be an algebra of operators such that its diagonal is $\mathfrak{A}$. Let $\mathcal{M}$ be a subspace of $\mathcal{F}$ which is invariant under $\mathcal{R}$. Then $\mathcal{P}_\mathcal{M} \in \mathfrak{A}'$.

**Proof.** Suppose that $A$ is a self-adjoint element of $\mathfrak{A}$. Then we have

$$AP_\mathcal{M} = P_\mathcal{M}AP_\mathcal{M} = P_\mathcal{M}A^*P_\mathcal{M} = (P_\mathcal{M}AP_\mathcal{M})^* = (AP_\mathcal{M})^* = P_\mathcal{M}A.$$ 

Since $*$-algebra $\mathfrak{A}$ is generated by the self-adjoint elements in itself, we have $P_\mathcal{M} \in \mathfrak{A}'$.

**Lemma 3.** Let $\mathcal{H}$ be a complete nest of subspaces of a Hilbert space $\mathcal{F}$, and let $\mathcal{B}$ be the diagonal of nest algebra $\mathcal{A}_\mathcal{H}$ associated with $\mathcal{H}$. Then the core of $\mathcal{A}_\mathcal{H}$ is $\mathcal{A}$ (the commutant of $\mathfrak{A}$).

**Proof.** Let $\mathcal{B}$ be the core of $\mathcal{A}_\mathcal{H}$. Then by definition and Lemma 1, $\mathcal{B}$ is the von Neumann algebra generated by the set of projections $\{ P_\mathcal{M} | \mathcal{M} \in \mathcal{H} \}$. And by Lemma 2 $P_\mathcal{M} \in \mathfrak{A}'$ for each $\mathcal{M} \in \mathcal{H}$. Hence we have $\mathcal{B} \subseteq \mathfrak{A}'$.

If $B \in \mathcal{B}'$ and $B^* = B$, then $B$ commutes with $\mathcal{B}$ and, all the more, with each $P_\mathcal{M}(\mathcal{M} \in \mathcal{H})$. Hence $B \in \mathcal{A}_\mathcal{H}$ and $B \in \mathcal{A}_\mathcal{H} \cap \mathcal{A}_\mathcal{H}^* = \mathfrak{A}$. Since $\mathcal{B}'$ is generated by the self-adjoint operators in itself, we have $\mathcal{B}' \subseteq \mathfrak{A}$ and then $\mathcal{B} \supseteq \mathfrak{A}'$.

This completes the proof of the lemma.

The following result is a characterization of nest algebras in a sense.

**Theorem 4.** Let $\mathfrak{A}$ be a von Neumann algebra on a Hilbert space $\mathcal{F}$ such that its commutant $\mathfrak{A}'$ is abelian, and let $\mathcal{R}$ be an algebra of operators such that its diagonal is $\mathfrak{A}$, i.e., $\mathcal{R} \cap \mathcal{R}^* = \mathfrak{A}$. Then $\mathcal{R}$ is a nest algebra if and only if the following conditions are satisfied:

1° $\mathcal{R}$ is maximal with respect to the property of having $\mathfrak{A}$ as its diagonal;

2° $\mathfrak{A}'$ is the core of $\mathcal{R}$.

**Proof.** Suppose that $\mathcal{R}$ satisfies Conditions 1° and 2° and that $\mathcal{M}$ is any subspace of $\mathcal{F}$ which is invariant under $\mathcal{R}$. Then by Lemma
2 we have clearly \( P_{\mathcal{M}} \in \mathcal{U} \subseteq \mathcal{U}'' = \mathcal{A} \subseteq \mathcal{R} \). Therefore by virtue of ([1], Lemmas 2.3.2 and 2.3.3) and our Condition 1\(^{\circ}\), the family \( \mathcal{N} \) of the invariant subspaces under \( \mathcal{R} \) is a complete nest. Let \( \mathcal{A}_{\mathcal{N}} \) be the nest algebra associated with \( \mathcal{N} \). Then clearly \( \mathcal{R} \subseteq \mathcal{A}_{\mathcal{N}} \). Suppose that \( A \in \mathcal{A}_{\mathcal{N}} \cap \mathcal{A}_{\mathcal{N}}^{*} \) and \( A^{*} = A \). Then we have for each \( \mathcal{M} \in \mathcal{N} \),

\[
AP_{\mathcal{M}} = P_{\mathcal{M}} AP_{\mathcal{M}} = P_{\mathcal{M}} A^{*} P_{\mathcal{M}} = (P_{\mathcal{M}} A P_{\mathcal{M}})^{*} = (A P_{\mathcal{M}})^{*} = P_{\mathcal{M}} A.
\]

Hence by Condition 2\(^{\circ}\), \( A \in \mathcal{U}'' = \mathcal{A} \). Since a \(*\)-algebra is generated by the self-adjoint elements in itself, we have

\[
\mathcal{A}_{\mathcal{N}} \cap \mathcal{A}_{\mathcal{N}}^{*} \subseteq \mathcal{A}.
\]

On the other hand, it follows from the above mentioned property \( \mathcal{R} \subseteq \mathcal{A}_{\mathcal{N}} \) that we have

\[
\mathcal{A}_{\mathcal{N}} \cap \mathcal{A}_{\mathcal{N}}^{*} \supseteq \mathcal{U}.
\]

Hence \( \mathcal{A}_{\mathcal{N}} \cap \mathcal{A}_{\mathcal{N}}^{*} = \mathcal{U} \), and by virtue of Condition 1\(^{\circ}\), \( \mathcal{R} = \mathcal{A}_{\mathcal{N}} \).

Suppose conversely that \( \mathcal{R} \) is the nest algebra associated with a complete nest \( \mathcal{N} \). Then by Lemma 3, \( \mathcal{R} \) satisfies Condition 2\(^{\circ}\). Therefore it remains to prove that \( \mathcal{R} \) satisfies also Condition 1\(^{\circ}\). But the very same argument as one used by Kadison and Singer in proof of ([1], Theorem 3.1.1) gives the following result: Let \( \mathcal{S} \) be an algebra which contains \( \mathcal{R} \) and is maximal with respect to the property of having \( \mathcal{U} \) as its diagonal. Then we have for each \( \mathcal{S} \in \mathcal{S} \) and each \( \mathcal{M} \in \mathcal{N} \),

\[
(I - P_{\mathcal{M}})SP_{\mathcal{M}} = 0.
\]

We omit repeating of the argument. Hence \( S \in \mathcal{R} \) and then \( \mathcal{S} = \mathcal{R} \). This completes the proof of the theorem.

In the next we give another characterization of nest algebras.

**Theorem 5.** Let \( \mathcal{A} \) be a von Neumann algebra on a Hilbert space \( \mathcal{H} \) such that its commutant \( \mathcal{A}' \) is abelian, and let \( \mathcal{R} \) be a weakly closed subalgebra of \( \mathcal{L}(\mathcal{H}) \) which is maximal with respect to the property of having \( \mathcal{A} \) as its diagonal. Then \( \mathcal{R} \) is a nest algebra.

**Proof.** Let \( \mathcal{N} \) be the family of the subspaces of \( \mathcal{H} \) which are invariant under \( \mathcal{R} \). Then we proved, in proof of Theorem 4, that \( \mathcal{N} \) is a complete nest. Hence by virtue of ([2], Theorem 2) it will suffice to show that \( \mathcal{R} \) contains a maximal abelian self-adjoint algebra. Since \( \mathcal{A}' \subseteq \mathcal{A} \), there is a maximal abelian sub \(*\)-algebra \( \mathcal{B} \) of \( \mathcal{A} \) which contains \( \mathcal{A}' \), by Zorn's lemma. Let \( X \) be any operator in \( \mathcal{L}(\mathcal{H}) \) which commutes with each member \( B \) in \( \mathcal{B} \). Then clearly \( X \) commutes with each member in \( \mathcal{A}' \). Hence \( X \in \mathcal{A}''' = \mathcal{A} \). By maximality in \( \mathcal{A} \), \( X \in \mathcal{B} \). Then \( \mathcal{B} \) is a maximal abelian self-adjoint algebra in \( \mathcal{L}(\mathcal{H}) \). This completes the proof of the theorem.

**Corollary.** Each weakly closed maximal triangular algebra is hyperreducible.

**Remark.** In the preparation of this paper we were informed by

References