75. *l^p-spaces over Banach Spaces and an Application**)

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1. Polynomial maps (more generally, analytic maps) of Banach spaces have been studied by several authors [1], [2]. In this note we shall study a polynomial map by factoring into a composition of a linear map and a map looks like the exponential map. For this purpose we shall define a new Banach space l_s^pE over a Banach space E. This treatment of polynomial maps enable us to reduce some problems on polynomial maps to the well known facts on linear maps. As a simple example we shall give a proof of the regularity theorem for a solution of semi-linear polynomial elliptic differential equation.

Let E be a real or complex Banach space with norm $\| \ \|$. We shall denote by $E^{\otimes n}$ the completion of the n^{th} tensor power of E with respect to the projective topology. The norm $\| \ \|_n$ of x in $E^{\otimes n}$ is defined by $\|x\|_n = \inf\{\sum \|x_1^{(i)}\| \cdots \|x_n^{(i)}\| \,|\, x = \sum x_1^{(i)} \otimes \cdots \otimes x_n^{(i)}\}$.

Let $l^pE (1 \leqq p < \infty)$ be the completion of the (algebraic) vector space $\bigoplus_{n=1}^{\infty} E^{\otimes n}$ with the l^p-norm $\| \ \|_{l^p}$ defined by $\|x\|_{l^p}^p = \sum \|x_n\|_n^p$, for $x = \sum x_n$, $x_n \in E^{\otimes n}$. Thus an element x of l^pE can be written as an infinite sum $x = \sum x_n$ of elements $x_n \in E^{\otimes n}$. It is clear that l^pE is a Banach space. As usual, we have $l^pE \subset l^qE$ if $p \leqslant q$ and the inclusion is continuous. Note that if $E = \boldsymbol{R}$ or $\boldsymbol{C}$, l^pE is canonically isomorphic to the ordinary l^p-space. If E is a separable Hilbert space, we can define an inner product in l^2E which then is again a Hilbert space.

Let $E_s^{\otimes n}$ be the subspace of symmetric elements of $E^{\otimes n}$, the Banach subspace l_s^pE of l^pE is defined to be the completion of $\bigoplus_{n=1}^{\infty} E_s^{\otimes n}$ with the l^p-norm.

For two Banach spaces E and F, the following proposition is easily proved.

Proposition 1. (1) *$l^p(E \oplus F) \subset l^pE \oplus l^pF$, $l_s^p(E \oplus F) \subset l_s^pE \oplus l_s^pF$.* (2) *$l^p(E \otimes F) \cong l^pE \otimes l^pF$, $l_s^p(E \otimes F) \cong l_s^pE \otimes l_s^pF$.* (3) *If E is finite dimensional and $p > 1$, $(l^pE)' \cong l^qE'$ and $(l_s^pE)' \cong l_s^qE'$, where E' is the dual space of E and $\dfrac{1}{p} + \dfrac{1}{q} = 1$.*

A Banach space E is a Banach algebra if there is a continuous

*) Dedicated to Professor Atuo Komatu on his 60th birthday.

linear map $\mu: E\otimes E\to E$ such that $\|\mu\|\leqq 1$ and $\mu(\mu\otimes id_E)=\mu(id_E\otimes\mu)$. Then we can define a linear map $\mu_n: E^{\otimes n}\to E$, for $n\geqq 3$, such that $\|\mu_n\|\leqq 1$ and $\mu(\mu_{n-i+1}\otimes\mu_i)=\mu(\mu_n\otimes id_E)$ for $1\leqq i\leqq n$ where $\mu_2=\mu$ and $\mu_1=id_E$. Let $m: l^pE\to E$ be a map defined by $m(\sum x_n)=\sum\mu_n(x_n)$, $x_n\in E^{\otimes n}$, then m is a continuous linear map with $\|m\|\leqq 1$. We also define a continuous linear map $m_s: l_s^pE\to E$ by $m_s=m\,|\,l_s^pE$.

Let E and F be Banach spaces and $f_i: E\to F$ $(i=1,\cdots,n)$ be continuous linear maps, then a continuous linear map $f_1\otimes\cdots\otimes f_n: E^{\otimes n}\to F^{\otimes n}$ is defined by $(f_1\otimes\cdots\otimes f_n)(\sum x_1^{(i)}\otimes\cdots\otimes x_n^{(i)})=\sum f_1(x_1^{(i)})\otimes\cdots\otimes f_n(x_n^{(i)})$. In fact we have $\|f_1\otimes\cdots\otimes f_n\|\leqq\|f_1\|\cdots\|f_n\|$. If $f: E\to F$ is a linear map with $\|f\|\leqq 1$, we can define a linear map $l^pf: l^pE\to l^pF$ by $(l^pf)(\sum x_n)=\sum f^{\otimes n}(x_n)$, $x_n\in E^{\otimes n}$, where $f^{\otimes n}=f\otimes\cdots\otimes f$ (n copies). Then we have $\|l^pf\|=\|f\|$, and hence l^pf is continuous. It is easily seen that $(l^pf)(l_s^pE)\subset l_s^pF$, and $l^p(g\circ f)=l^pg\circ l^pf$ for linear maps $f: E\to F$ and $g: F\to G$ of Banach spaces with $\|f\|\leqq 1$ and $\|g\|\leqq 1$.

Let $U(E)$ be the group of linear isometries of E, and $l^pU(E)=\{l^pf\,|\,f\in U(E)\}$. Then we have

Proposition 2. *$l^pU(E)$ is a closed subgroup of the group $U(l_s^pE)$ of linear isometries of l_s^pE.*

Let $f: E\to F$ be a (not necessarily linear) map of Banach spaces. Then f is differentiable at $x_0\in E$ if there is a continuous linear map $df(x_0): E\to F$ such that $\lim_{v\to 0}(\|f(x_0+v)-f(x_0)-df(x_0)(v)\|_F)/\|v\|_E=0$. The k^{th} derivative $d^kf: E\to L_s^k(E,F)$ $(=L(E_s^{\otimes k},F))$ is defined inductively by $d^kf=d(d^{k-1}f)$, and f is of class C^k if d^kf is continuous. It is easily verified that $d^k(f_1\otimes\cdots\otimes f_n)=\Sigma d^{k_1}f_1\otimes\cdots\otimes d^{k_n}f_n$, where the sum ranges over all n-tuples $(k_1,\cdots,k_n)$ of non-negative integers with $k_1+\cdots+k_n=k$. If $\dim E=m<\infty$, the partial derivatives $D_if: E\to L(E,R)$ $(i=1,\cdots,m)$ is similarly defined and we have $D^\alpha(f_1\otimes\cdots\otimes f_n)=\sum D^{\alpha_1}f_1\otimes\cdots\otimes D^{\alpha_n}f_n$, where the sum ranges over all n-tuples of multi-indices $(\alpha_1,\cdots,\alpha_n)$ with $\alpha_1+\cdots+\alpha_n=\alpha$.

2. Let E be a Banach space. We define a map $e: E\to l_s^pE$, for any $p\geqq 1$, by $e(x)=\sum\frac{1}{n!}x^{\otimes n}$, $x\in E$. Then easily we have

Theorem 1. *The map $e: E\to l_s^pE$ is of class C^∞.*

A map $f: E\to F$ of Banach spaces is called a *polynomial map* if there is a continuous linear map $\varphi: l_s^pE\to F$, for some $p\geqq 1$, such that $f=\varphi\circ e$. By definition, a polynomial map is of class C^∞.

Let $P(E,F)$ be the vector space of polynomial maps from E to F.

Theorem 2. *If E admits a basis, then the map $e^*: L(l_s^pE,F)\to P(E,F)$ defined by $e^*(\varphi)=\varphi\circ e$, for $\varphi\in L(l_s^pE,F)$, is an isomorphism for any p, $1\leqq p<\infty$.*

Lemma. $\sum_{\sigma\in S_n}x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(n)}=\sum_{k=1}^{n-1}(-1)^k(\sum_{\sigma'}(x_{\sigma'(1)}+\cdots+x_{\sigma'(n-k)})^{\otimes n})$, *for $x_1, \cdots, x_n \in E$, where S_n is the n^{th} symmetric group and σ' ranges over all combinations of $(n-k)$ elements of the set $\{1, \cdots, n\}$.*

This Lemma is easily checked by a simple calculation.

Proof of Theorem 2. By definition, e^* is a homomorphism onto $P(E,F)$. Let $\{u_1, \cdots, u_n, \cdots\}$ be a basis for E, then $\{u_{i_1,\cdots,i_n} = \sum_{\sigma\in S_n} u_{i_{\sigma(1)}} \otimes\cdots\otimes u_{i_{\sigma(n)}} \mid i_1 \leqslant i_2 \leqslant \cdots \leqslant i_n\}$ forms a basis for $E_s^{\otimes n}$. Let $\varphi \in L(l_s^p E, F)$ be a map such that $\varphi(e(x))=0$ for any $x \in E$. Then for each base u_i of E and for any real $\lambda \neq 0$, we have $0=\varphi(e(\lambda u_i)) = \sum \frac{\lambda^n}{n!}\varphi(u_i^{\otimes n})$ so that $\frac{1}{\lambda}\varphi(e(\lambda u_i)) = \varphi(u_i) + \lambda\Phi(u_i) = 0$, hence $\varphi(u_i) = \lim_{\lambda\to 0}(-\lambda\Phi(u_i)) = 0$. Inductively, we assume that $\varphi(u_{i_1,\cdots,i_k})=0$ for any $u_{i_1,\cdots,i_k}$ with $k<n$. Then, by the above Lemma, for any $u_{i_1,\cdots,i_n} \in E_s^{\otimes n}$ and for any real $\lambda \neq 0$,

$$\begin{aligned}0 &= n!\varphi(e(\lambda \sum_{k=1}^{n-1}(-1)^k(\sum_{\sigma'} u_{i_{\sigma'(1)}} + \cdots + u_{i_{\sigma'(n-k)}})) \\ &= \lambda^n \varphi(u_{i_1,\cdots,i_n}) + \lambda^{n+1}\Phi(u_{i_1,\cdots,i_n}),\end{aligned}$$

hence $\varphi(u_{i_1,\cdots,i_n}) = \lim_{\lambda\to 0}(-\lambda\Phi(u_{i_1,\cdots,i_n}))=0$. This implies that $\varphi=0$ so that e^* is an isomorphism. q.e.d.

Remark. The assumption that E admits a basis can be removed.

We shall define a topology on $P(E,F)$ such that e^* is a homeomorphism, and call it the l^p-topology of $P(E,F)$.

We can imbed $E_n^s = \bigoplus_{k=1}^{n} E_s^{\otimes k}$ in $l_s^p E$ for each $1 \leqq p < \infty$, and then let $\hat{E}_n^s$ be the supplementary subspace in $l_s^p E$. A polynomial map $f = \varphi\circ e : E \to F$ is said to be of *degree* n if $\varphi(x)=0$ for $x \in \hat{E}_n^s$. The vector space $P_n(E,F)$ of polynomial maps of degree n from E to F is a subspace of $P(E,F)$. We have $P_n(E,F) \subset P_m(E,F)$ if $n \leqq m$ and $P_1(E,F)$ is canonically isomorphic to $L(E,F)$. For three Banach spaces E, F and G, we have

Proposition 3. *$P_m(F,G)\circ P_n(E,F) \subset P_{mn}(E,G)$ and $L(F,G)\circ P(E,F) \subset P(E,G)$.*

It does not hold that $P(F,G)\circ L(E,F) \subset P(E,G)$, but if $f: E\to F$ is a linear map with $\|f\| \leqq 1$ then we have $P(F,G)\circ f \subset P(E,G)$.

3. In this section we shall freely use the methods and results of Palais [3; Chap. IV, VIII, XI].

Let M be a (finite dimensional) compact C^∞ manifold without boundary and with a fixed strictly positive smooth measure. For a (finite dimensional) hermitian vector bundle ξ over M, we define a Hilbert vector bundle $l_s^2\xi$ over M by $l_s^2\xi = \bigcup_{x\in M} l_s^2\xi_x$ with the group $l^2U(\xi)$ where $U(\xi)$ is the group of unitary transformations of ξ. Thus the structure of $l_s^2\xi$ depends on the hermitian structure of ξ. The map $e_x : \xi_x \to l_s^2\xi_x$, $x \in M$, induces a C^∞ bundle map $e: \xi \to l_s^2\xi$.

A bundle map $f:\xi\to\eta$ is a *polynomial map* if there is a bundle homomorphism $\varphi: l_s^2\xi\to\eta$ such that $f=\varphi\circ e$. Let $\mathrm{Pol}(\xi,\eta)$ be the vector space of polynomial maps from ξ to η, then by Theorem 2 we have an isomorphism $e^*: \mathrm{Hom}(l_s^2\xi,\eta)\to\mathrm{Pol}(\xi,\eta)$.

Let $C^\infty(\xi)$ be the vector space of (global) C^∞ sections of the bundle ξ. For two hermitian vector bundles ξ and η over M, $L(\xi,\eta)$ is the vector bundle of linear maps $\xi_x\to\eta_x$, for each $x\in M$, such that $C^\infty L(\xi,\eta)=\mathrm{Hom}(\xi,\eta)$. Similarly $P(\xi,\eta)$ is defined to be the vector bundle such that $C^\infty P(\xi,\eta)=\mathrm{Pol}(\xi,\eta)$. We have again a bundle isomorphism $e^*: L(l_s^2\xi,\eta)\to P(\xi,\eta)$.

A map $f: C^\infty(\xi)\to C^\infty(\eta)$ is said to be *polynomial* (in narrow sense) if there is a linear map $\varphi: C^\infty(l_s^2\xi)\to C^\infty(\eta)$ such that $f=\varphi\circ\bar{e}$ where $\bar{e}: C^\infty(\xi)\to C^\infty(l_s^2\xi)$ is the map induced by $e:\xi\to l_s^2\xi$.

Let $A(\xi,\eta)$ be a vector space of linear operators from ξ to η, that is, an element of $A(\xi,\eta)$ is a linear map $T: C^\infty(\xi)\to C^\infty(\eta)$, then we define a vector space $PA(\xi,\eta)$ of polynomial operators from ξ to η by $PA(\xi,\eta)=\{T: C^\infty(\xi)\to C^\infty(\eta)\mid T=\mathcal{T}\circ\bar{e}$ for some $\mathcal{T}\in A(l_s^2\xi,\eta)\}$. In this case the map $\bar{e}^*: A(l_s^2\xi,\eta)\to PA(\xi,\eta)$ is only an epimorphism in general.

Let $T^*(M)$ be the cotangent bundle of M and $T'(M)$ be the bundle $T^*(M)$ with the zero section removed. Let $\pi: T'(M)\to M$ be the projection and ξ be a vector bundle over M, then $\pi^*(\xi)$ is a vector bundle over $T'(M)$ and $\mathrm{Pol}(\pi^*\xi,\pi^*\eta)$ consists of functions σ on $T'(M)$ such that $\sigma(v,x)$ is a polynomial map of ξ_x into η_x. We define a vector space $\mathrm{P\,Smbl}_k(\xi,\eta)$ by $\mathrm{P\,Smbl}_k(\xi,\eta)=\{\sigma\in\mathrm{Pol}(\pi^*\xi,\pi^*\eta)\mid\sigma(\rho v,x)=\rho^k\sigma(v,x)$ if $\rho>0\}$. Again we have an isomorphism $e^*: \mathrm{Smbl}_k(l_s^2\xi,\eta)\to\mathrm{P\,Smbl}_k(\xi,\eta)$.

In [3], several vector spaces of linear operators are defined for hermitian vector bundles over M. These are $\mathrm{OP}_k(\xi,\eta)$, $\mathrm{Int}_k(\xi,\eta)$ and $\mathrm{Diff}_k(\xi,\eta)$ etc. For precise definitions and properties of these spaces we refer to [3]. From these we can define corresponding spaces of polynomial operators, that is, $\mathrm{POP}_k(\xi,\eta)$, $\mathrm{P\,Int}_k(\xi,\eta)$ and $\mathrm{P\,Diff}_k(\xi,\eta)$ etc.

In [3; Chap. XI], it is proved that the sequence $0\to\mathrm{OP}_{k-1}(\xi,\eta)\to\mathrm{Int}_k(\xi,\eta)\xrightarrow{\sigma_k}\mathrm{Smbl}_k(\xi,\eta)\to 0$ is exact for any hermitian vector bundles ξ, η over M where $\sigma_k: \mathrm{Int}_k(\xi,\eta)\to\mathrm{Smbl}_k(\xi,\eta)$ is the symbol map. Although $\bar{e}^*$ are only epimorphisms we have

Proposition 4. *The sequence* $0\to\mathrm{POP}_{k-1}(\xi,\eta)\to\mathrm{P\,Int}_k(\xi,\eta)\xrightarrow{\tilde{\sigma}_k}\mathrm{P\,Smbl}_k(\xi,\eta)\to 0$ *is exact for any hermitian vector bundles* ξ, η *over* M.

Since $\mathrm{Smbl}_k(\xi,\eta)\subset\mathrm{P\,Smbl}_k(\xi,\eta)$, we call a polynomial operator $T\in\mathrm{P\,Int}_k(\xi,\eta)$ *semilinear* if $\tilde{\sigma}_k(T)$ is contained in $\mathrm{Smbl}_k(\xi,\eta)$.

A semilinear polynomial operator $T \in \mathrm{P\,Int}_k(\xi, \eta)$ is called k^{th} *order elliptic* if $\tilde{\sigma}_k(T)(v, x)$ maps ξ_x isomorphically onto η_x for all $(v, x) \in T'(M)$. It is proved in [3] that if a linear operator $S \in \mathrm{Int}_k(\xi, \eta)$ is k^{th} order elliptic then there exists $S' \in \mathrm{Int}_{-k}(\eta, \xi)$ which is $-k^{\text{th}}$ order elliptic such that $\sigma_{-k}(S') = \sigma_k(S)^{-1}$, $S'S - I_\xi \in \mathrm{OP}_{-1}(\xi, \xi)$ and $SS' - I_\eta \in \mathrm{OP}_{-1}(\eta, \eta)$. Similarly we have

Proposition 5. *If a semilinear polynomial operator $T \in \mathrm{P\,Int}_k(\xi, \eta)$ is k^{th} order elliptic then there is a linear operator $T' \in \mathrm{Int}_{-k}(\eta, \xi)$ which is $-k^{\text{th}}$ order elliptic such that $\sigma_k(T') = \tilde{\sigma}_k(T)^{-1}$ and $T'T - I_\xi \in \mathrm{POP}_{-1}(\xi, \xi)$.*

Now, analogously to Theorem 5 of [3; Chap. XI], we give a proof to the (well-known) theorem of regularity of a solution of semilinear elliptic polynomial equation.

Theorem 3. *Let T be a semilinear elliptic polynomial operator in $\mathrm{P\,Int}_k(\xi, \eta)$. If $f \in H^{-\infty}(\xi)$ and $\bar{T}f \in H^r(\eta)$ then $f \in H^{r+k}(\xi)$, where $H^k(\xi)$ is the Sobolev spaces on $C^\infty(\xi)$ and $\bar{T}: H^{-\infty}(\xi) \to H^{-\infty}(\eta)$ is the extension of T. (For a precise definition, see [3])*

Proof. Since $H^{-\infty}(\xi) = \cup H^m(\xi)$, $f \in H^m(\xi)$ for some m. By induction, it suffices to prove that if $m < r + k$ then $f \in H^{m+1}(\xi)$. By the above Proposition, there is a linear operator $T' \in \mathrm{Int}_{-k}(\eta, \xi)$ which is $-k^{\text{th}}$ order elliptic such that $T'T - I_\xi \in \mathrm{POP}_{-1}(\xi, \xi)$, so that $(\bar{T}'\bar{T}f - f) \in H^{m+1}(\xi)$. On the other hand, since $\bar{T}f \in H^r(\eta)$, $\bar{T}'\bar{T}f \in H^{r+k}(\xi) \subset H^{m+1}(\xi)$. Hence we have $f \in H^{m+1}(\xi)$. q.e.d.

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