

94. On Representations of Tensor Products of Involutive Banach Algebras

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In this paper we want to give a complementary result to the previous paper [2]. The aimed theorem states that; if each of involutive Banach algebras A and B has approximating identity and has a faithful representation, then any representation π of the algebraic tensor product $A \odot B$ of A and B is *subcross* in the sense of [1], in other words, it is satisfied that

$$\|\pi(x \otimes y)\| \leq \|x\| \|y\| \quad \text{for } x \in A \text{ and } y \in B.$$

This means that

$$\|\pi(t)\| \leq \|t\|_\gamma \quad \text{for } t \in A \odot B,$$

where $\|\cdot\|_\gamma$ denotes the γ -norm on $A \odot B$, and that there are representations π^1, π^2 of A, B , called the *restrictions* of π on A, B , respectively, on the representation space of π such that

$$\pi(x \otimes y) = \pi^1(x)\pi^2(y) = \pi^2(y)\pi^1(x) \quad \text{for } x \in A \text{ and } y \in B.$$

These assertions seem to be important in investigations of algebraic tensor products of involutive Banach algebras from standpoints of C^* -algebras.

As Professor A. Guichardet kindly pointed out by his private letter, the arguments of Lemma 1 and Theorem 1 in [2] are lacking in exactness. A part of the following is devoted to remove their inexactness. It is done by imposing a natural condition upon involutive Banach algebras considered. The author wishes to take this opportunity to deeply thank Professor Guichardet.

1. Preliminaries. An involutive algebra means an algebra over the complex number field C with an involution always denoted by $*$. Given an involutive algebra A , the adjunction A_1 of the identity to A means A itself when A has an identity, the involutive algebra of all formal sums $u = x + \lambda$ of $x \in A$ and $\lambda \in C$ when A has no identity. A representation of an involutive algebra means its involution-preserving homomorphism into the algebra of bounded linear operators on a complex Hilbert space.

An involutive Banach algebra means an involutive algebra equipped with a norm under which it is a Banach algebra and its involution is isometric. When A is an involutive Banach algebra, as

is well-known, A_1 is an involutive Banach algebra under a norm to which the initial norm on A is extended.

A norm $\| \cdot \|$ on an involutive algebra A is called a C^* -norm if it makes A a normed algebra and it satisfies that

$$\|x^*x\| = \|x\|^2 \quad \text{for } x \in A.$$

We start with the next

Lemma 1. *Let A be an involutive algebra, I an ideal in A closed under the involution, $\| \cdot \|$ a C^* -norm given on I . If for each $x \in A$ the number set $\{\|xy\| : y \in I \text{ and } \|y\| \leq 1\}$ is bounded, then, $\| \cdot \|^\wedge$ defined by*

$$\|x\|^\wedge = \sup_{y \in I \text{ with } \|y\| \leq 1} \|xy\|, \quad x \in A$$

becomes a semi-norm on A , coincides with $\| \cdot \|$ on I and satisfies that

$$\|xy\|^\wedge \leq \|x\|^\wedge \|y\|^\wedge$$

and

$$\|x^*x\|^\wedge = \|x\|^{\wedge 2}$$

for x and y in A . Moreover, it is a C^* -norm if and only if the left annihilator $L_A(I)$ of I contains no other element than 0.

Proof. We show only that $\|x^*x\|^\wedge = \|x\|^{\wedge 2}$ because the remainders are obvious. Given a positive number η smaller than 1, there is an element y in I with $\|y\| \leq 1$ such that

$$\eta \|x\|^\wedge \leq \|xy\|.$$

Therefore,

$$\eta^2 \|x\|^{\wedge 2} \leq \|xy\|^2 = \|y^*x^*xy\| \leq \|x^*xy\| \leq \|x^*x\|^\wedge.$$

Thus we know that

$$\|x\|^{\wedge 2} \leq \|x^*x\|^\wedge \leq \|x^*\|^\wedge \|x\|^\wedge.$$

The desired formula follows at once from this inequality and the proof is completed.

We are interested in the roof extension $\| \cdot \|^\wedge$ of $\| \cdot \|$ introduced in such a way. If A is a C^* -algebra with a norm $\| \cdot \|$, B a dense subalgebra of A , I an ideal in B , each of them is closed under the involution and the closure J of I in A satisfies that $L_A(J) = \{0\}$, then, denoting by $\| \cdot \|_1$ the restriction of $\| \cdot \|$ on I , $\| \cdot \|_1^\wedge$ is a C^* -norm on B and

$$\|x\|_1^\wedge = \|x\| \quad \text{for } x \in B.$$

In fact, $\| \cdot \|_2$ being the restriction of $\| \cdot \|$ on J , $\| \cdot \|_2^\wedge$ becomes a C^* -norm on A by Lemma 1, so it turns out to coincide with $\| \cdot \|$ and for each $x \in B$,

$$\|x\|_1^\wedge = \sup_{y \in I \text{ with } \|y\| \leq 1} \|xy\| = \sup_{y \in J \text{ with } \|y\| \leq 1} \|xy\| = \|x\|_2^\wedge = \|x\|.$$

Let A and B be involutive algebras, I, J ideals in A, B , respectively, closed under the involutions. Then by a short consideration we know that $L_A(I) = \{0\}$ and $L_B(J) = \{0\}$ implies $L_{A \otimes B}(I \odot J) = \{0\}$ and vice versa. Thus it can be remarked that; if A and B are C^* -

algebras, I, J ideals in A, B , respectively, closed under the involutions, each left annihilator contains no other element than 0 and $\|\cdot\|$ the restriction on $I \odot J$ of the α -norm $\|\cdot\|_\alpha$ on $A \odot B$, then $\|\cdot\|^\wedge$ coincides with $\|\cdot\|_\alpha$. This fact is known immediately from the minimality of the α -norm (see [1] and [2]).

2. Compatibility of C^* -norms. Lemma 1 and Theorem 1 in [2] becomes valid by being imposed a condition that involutive Banach algebras under consideration have approximating identities and they are restated as

Lemma 2. *Let A and B be involutive Banach algebras with approximating identities, then any C^* -norm $\|\cdot\|$ on $A \odot B$ is subcross, in other words, it satisfies that*

$$\|x \otimes y\| \leq \|x\| \|y\| \quad \text{for } x \in A \text{ and } y \in B,$$

and it can be extended to a C^* -norm on $A_1 \odot B_1$.

Here an approximating identity of a Banach algebra A means a net $\{u_\xi\}$ lying in the closed unit ball of A such that

$$\lim_{\xi} \|u_\xi x - x\| = \lim_{\xi} \|x u_\xi - x\| = 0 \quad \text{for each } x \in A.$$

The author does not know whether the existence of approximating identities is inevitable for the conclusions above. It, however, gives no unpleasant effects to other statements in [2].

Proof. First we prove that the roof extension onto $A_1 \odot B_1$ of the given $\|\cdot\|$ is defined and becomes a C^* -norm. Because A and B have approximating identities, we know that $L_A(A) = \{0\}$ and that $L_B(B) = \{0\}$. So $L_{A_1}(A) = \{0\}$ and $L_{B_1}(B) = \{0\}$. Therefore, $L_{A_1 \odot B_1}(A \odot B) = \{0\}$. Thus, on that account of Lemma 1, it is sufficient for our purpose to show that

$$(*) \quad \sup_{v \in A \odot B \text{ with } \|v\| \leq 1} \|uv\| < \infty \quad \text{for each } u \in A_1 \odot B_1.$$

Given a positive element h in A and a state f of the $\|\cdot\|$ -product $D = A \otimes_{\|\cdot\|} B$ of A and B , the C^* -algebra obtained as the completion of $A \odot B$ with respect to $\|\cdot\|$, the functional $g_{h,f}$ on B defined by

$$g_{h,f}(y) = f(h \otimes y) \quad \text{for } y \in B$$

turns out to be continuous from a theorem of Varopoulos which asserts that any positive functional on an involutive Banach algebra with approximating identity is continuous (see [4]). Moreover, since

$$\sup_{f \in S} |g_{h,f}(y)| \leq \|h \otimes y\| < \infty$$

for each y in B , where S denotes the set of all states of D , the number set $\{\|g_{h,f}\| : f \in S\}$ is bounded from the uniform boundedness principle. We denote by K_h the positive square root of its supremum.

Let $x \in A$ and $y \in B$, then we know that

$$\|x \otimes y\| = \|x^* x \otimes y^* y\|^{1/2} = \sup_{f \in S} |g_{x^* x, f}(y^* y)|^{1/2} \leq K_{x^* x} \|y\|;$$

and by an analogous argument that

$$\|x \otimes y\| \leq L_{y^*y} \|x\|,$$

where L_{y^*y} is a positive number depends only on y . These facts show that the mapping $(x, y) \mapsto \|x \otimes y\|$ is separately continuous on $A \times B$.

Let next $v \in A \odot B$ and $\|v\| \leq 1$, then

$$\|(x \otimes 1)v\| = \lim_{\alpha} \|(x \otimes e_{\alpha})v\| \leq \lim_{\alpha} \|x \otimes e_{\alpha}\| \leq K_{x^*x},$$

$\{e_{\alpha}\}$ being an approximating identity of B , and similarly,

$$\|(1 \otimes y)v\| \leq L_{y^*y}.$$

Therefore, if $A_1 \odot B_1 \ni u = t + x \otimes 1 + 1 \otimes y + \lambda 1 \otimes 1$, $t \in A \odot B$, $x \in A$, $y \in B$ and $\lambda \in C$, then

$$\|uv\| \leq \|tv\| + \|(x \otimes 1)v\| + \|(1 \otimes y)v\| + \|\lambda v\| \leq \|t\| + K_{x^*x} + L_{y^*y} + |\lambda|$$

for each $v \in A \odot B$ with $\|v\| \leq 1$ and we have (*).

That $\|\cdot\|^\wedge$ is a C^* -norm on $A_1 \odot B_1$ is used to show that $\|\cdot\|$ is subcross. Since the mappings $x \mapsto x \otimes 1$ and $y \mapsto 1 \otimes y$ of A , B , respectively, into the $\|\cdot\|^\wedge$ -product of A_1 and B_1 are involution-preserving isomorphisms, we know that

$$\|x \otimes 1\|^\wedge \leq \|x\|$$

and that

$$\|1 \otimes y\|^\wedge \leq \|y\|$$

for any $x \in A$ and $y \in B$. Thus at last we know that

$$\|x \otimes y\| = \|(x \otimes 1)(1 \otimes y)\|^\wedge \leq \|x \otimes 1\|^\wedge \|1 \otimes y\|^\wedge \leq \|x\| \|y\|$$

for any $x \in A$ and $y \in B$ and the proof is completed.

Lemma 2 makes us conclude that any C^* -norm on the algebraic tensor product of C^* -algebras is cross, because the minimal C^* -norm $\|\cdot\|_\alpha$ is cross as was shown in [3].

3. Compatibility of representations.

Theorem. *Let A and B be involutive Banach algebras, each of them have approximating identity and have a faithful representation. If π is a representation of $A \odot B$, then*

(a) $\|\pi(x \otimes y)\| \leq \|x\| \|y\|$ for $x \in A$ and $y \in B$,

(b) $\|\pi(t)\| \leq \|t\|_r$ for $t \in A \odot B$, and

(c) *there are representations π^1, π^2 of A, B , respectively, on the representation space of π such that*

$$\pi(x \otimes y) = \pi^1(x)\pi^2(y) = \pi^2(y)\pi^1(x) \quad \text{for } x \in A \text{ and } y \in B.$$

Proof. Let ρ and σ be faithful representations of A and B respectively, then $\pi \oplus (\rho \otimes \sigma)$ becomes a faithful representation of $A \odot B$. Thus $\|(\pi \oplus (\rho \otimes \sigma))(\cdot)\|$ defines a C^* -norm on $A \odot B$. Therefore from Lemma 2 for any $x \in A$ and $y \in B$,

$$\|\pi(x \otimes y)\| \leq \|(\pi \oplus (\rho \otimes \sigma))(x \otimes y)\| \leq \|x\| \|y\|$$

and (a) is proved. (b) follows from the definition of the γ -norm and (c) is a direct application of Remark 1 in [1] (see also Lemma 2 in [2]). Then the proof is completed.

From the above theorem it is concluded when A and B are

C^* -algebras that

$$\|t\|_\nu = \sup_{\pi \text{ representation of } A \otimes B} \|\pi(t)\| \quad \text{for } t \in A \otimes B$$

and that any representation π of $A \otimes B$ is continuous with respect to the ν -norm. In connection with this fact it is remarked that any representation of an involutive algebra is continuous with respect to its maximal C^* -norm if it exists.

References

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