122. On Quasi-Souslin Space and Closed Graph Theorem

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L. Schwartz defined *Souslin space* as any continuous image of a complete separable metric space and a generalized closed graph theorem is obtained in [1] and [2] for this class of spaces. Here we consider a slightly wider class of topological spaces, namely *quasi-Souslin spaces*, and prove a closed graph theorem extending the method in [2].

A filter Φ is said to be *S-filter* if Φ has a countable basis $\{S_n\}$ such that $\bigcap_n S_n=\phi$.

A Housdorff topological space E is called a *quasi-Souslin space*, if there exists a sequence of S-filters Φ_n $(n=1,2,\cdots)$ such that every ultrafilter Ψ with $\Psi \not\supset \Phi_n$ $(n=1,2,\cdots)$ converges in E. In the sequel Φ_n are called *defining filters for E*.

Let A be a subset of a set E and Φ a filter in E. We say that A is *disjoint from* Φ if there is B in Φ such that $A\cap B=\phi$. If A is not disjoint from Φ, we denote the filter $\{A\cap B \mid B\in\Phi\}$ in A by Φ_A. We identify the filter Φ_A in A with the filter Φ in E if $A\in\Phi$. Let φ be a mapping from a set E into a set F and Φ, Ψ filters in E, F respectively. $\varphi(\Phi)$, the *image of* Φ *by* φ, is defined as the filter generated by $\{\varphi(A) \mid A\in\Phi\}$. When $\varphi(E)$ is not disjoint from Ψ, $\varphi^{-1}(\Psi)$, the *inverse image of* Ψ *by* φ is defined as the filter generated by $\{\varphi^{-1}(A) \mid A\in\Psi\}$.

A subset A of topological space E is said to be *everywhere second category in E*, if any non-void intersection $U\cap A$ with an open set U in E is second category. As well known, if A is second category, the set $O(A)$ of all the elemets x in E for which $A\cap V$ is second category for every neighbourhood V of x is not empty and $O(A)\cap A$ is everywhere second category in E.

First we show that the class of quasi-Souslin spaces, as in the case of Souslin spaces, is closed by the following operations:

(1) *The image $E=\varphi(F)$ of a quasi-Souslin space F by a continuous mapping φ is quasi-Souslin.*

(2) *The closed subspace E of a quasi-Souslin space F is quasi-Souslin.*

(3) *The product space $E=\prod_n E_n$ of quasi-Souslin spaces E_n $(n=1,2,\cdots)$ is quasi-Souslin.*

(4) *The inductive limit E of quasi-Souslin spaces E_n $(n=1,2,\cdots)$ is quasi-Souslin.*

Proof of (1). Suppose F is quasi-Souslin with respect to S-filters Φ_n $(n=1,2,\cdots)$. Let D be a subset of F for which $\varphi(D)=\varphi(F)$ and φ is one-to-one on D. For each n such that D is not disjoint from Φ_n, $(\Phi_n)_D$ is an S-filter in D and $\varphi\{(\Phi_n)_D\}$ is an S-filter in E. Let Ψ be any ultrafilter in E such that $\Psi \not\supset \varphi\{(\Phi_n)_D\}$ for every n for which $(\Phi_n)_D$ exists, then $\{\varphi^{-1}(\Psi)\}_D \not\supset (\Phi_n)_D$ for every n. Therefore, if we choose an ultrafilter Ω in F such that $\{\varphi^{-1}(\Psi)\}_D \subset \Omega$ then $\varphi(\Omega)=\Psi$ and $\Omega \not\supset \Phi_n$ for every n $(n=1,2,\cdots)$. Since F is quasi-Souslin, there exists x in F such that Ω converges to x and hence Ψ converges to $\varphi(x)$ in E by virtue of the continuity of φ. Thus we have proved that E is quasi-Souslin with defining filters $\varphi\{(\Phi_n)_D\}$.

Proof of (2). Let Φ_n $(n=1,2,\cdots)$ be defining S-filters for F. For each n such that E is not disjoint from Φ_n, $(\Phi_n)_E$ is an S-filter in E. Then E is a quasi-Souslin space with respect to defining filters $(\Phi_n)_E$.

Proof of (3). For each n, let $\Phi_{n,m}$ $(m=1,2,\cdots)$ be defining S-filters for E_n and p_n projections from E to E_n. Then E is a quasi-Souslin space with defining filters $p_n^{-1}(\Phi_{n,m})$ $(n,m=1,2,\cdots)$.

Proof of (4). Suppose $E=\bigcup_n E_n$ with the including mappings f_n of E_n into E. For each n, let $\Phi_{n,m}$ $(m=1,2,\cdots)$ be defining filters for E_n and Φ the filter generated by $E\sim E_n$ $(n=1,2,\cdots)$. Then E is quasi-Souslin with defining filters:

$$\Phi \text{ and } f_n(\Phi_{n,m}) \ (n,m=1,2,\cdots).$$

The fact that every Souslin space is quasi-Souslin space is a consequence of (1) and the fact that every complete separable metric space is quasi-Souslin space. Let us prove this. Let E be a complete separable metric space, d the distance function in E, and D a dense countable subset in E. Put $U(x;\varepsilon)=\{y\in E \mid d(x,y)<\varepsilon\}$. Let Φ_n be the filter generated by the complements of all the finite union of $U\left(x;\frac{1}{n}\right)$, x being in D. Then Φ_n is S-filter in E and we show that E is a quasi-Souslin space with defining filters Φ_n $(n=1,2,\cdots)$. Let Ψ be an ultrafilter in E such that $\Psi \not\supset \Phi_n$ for every n, then there exists a sequence x_n of elements in D such that $\Psi \ni U\left(x_n;\frac{1}{n}\right)$. Then Ψ is a Cauchy filter in E and, since E is complete, Ψ converges.

Remark. Every compact Housdorff topological space and more generally every Housdorff topological space which is covered by countable number of compact subsets is quasi-Souslin. Reflexive non-separable Banach space with weak topology is an example of quasi-Souslin spaces which are not Souslin.

To prove the closed graph theorem for quasi-Souslin space, we make use of the following two Lemmas.

Lemma 1. *Let φ be a mapping from a topological space X into a regular topological space Y. Let A_i $(i=1,2,\cdots)$ be a dense subset of X such that $A_i \supset A_{i+1}$ for each i $(i=1,2,\cdots)$. Let Φ_x be the filter generated by all A_i $(i=1,2,\cdots)$ together with all the neighbourhoods of x in X. Then if $\varphi(\Phi_x)$ converges to $\varphi(x)$ for every $x \in X$, then φ is continuous.*

Proof. Let U be a neighbourhood of $x \in X$. Then for every $y \in U$, Φ_y converges to y and, by assumption, $\varphi(\Phi_y)$ converges to $\varphi(y)$ so $\varphi(y) \in \varphi(U \cap A_i)^-$, and hence $\varphi(U) \subset \varphi(U \cap A_i)^-$. Here $\varphi(U \cap A_i)^-$ denotes the closure of $\varphi(U \cap A_i)$. By virtue of the regularity of Y, for any neighbourhood V of $\varphi(x)$, there exists a closed neighbourhood W of $\varphi(x)$ such that $V \supset W$. Since $\varphi(\Phi_x)$ converges to $\varphi(x)$, there exists a neighbourhood U of x and i such that $\varphi(U \cap A_i) \subset V$. Then we have $\varphi(U) \subset \varphi(U \cap A_i)^- \subset V \subset W$. Therefore φ is continuous at x.

Lemma 2. *Let F be a linear topological space, and let D be a subset of F such that $F \sim D$, the complement of D in F, is first category in F, then every linear mapping φ of F into a linear topological space E which is continuous on D is continuous.*

Proof. For a given neighbourhood V of 0 in E, we show that there exists a neighbourhood U of 0 in F such that $\varphi(U) \subset V$. Let W be a neighbourhood of 0 in E such that $W - W \subset V$. Then, for a fixed element $x \in D$, by the continuity of φ on D, we can find U such that $\varphi(D \cap (x+U)) \subset W + \varphi(x)$. Let y be in U. As $x+U$ is a neighbourhood of x and $U-y$ is a neighbourhood of 0, $(x+U) \cap (x+U-y) \neq \phi$. As $F \sim D$ is first category, we have $\{(x+U) \cap D\} \cap \{((x+U) \cap D) - y\} \neq \phi$. Let z be in $\{(x+U) \cap D\} \cap \{((x+U) \cap D) - y\}$, then $z \in (x+U) \cap D$ and $z+y \in (x+U) \cap D$, and hence both $\varphi(z)$ and $\varphi(z+y)$ are in $\varphi(x)+W$. Therefore $\varphi(y) = \varphi(z+y) - \varphi(z) \in W - W$, showing $\varphi(U) \subset V$.

Now we prove

Theorem. *Every mapping φ with graph closed from a topological space F which is everywhere second category into a regular quasi-Souslin space E, is continuous on a subset D such that $F \sim D$ is first category.*

Proof. There exists a sequence of subsets A_i $(i=1,2,\cdots)$ of F such that, for each i, A_i is everywhere second category in F, $A_i \supset A_{i+1}$ and for every x in A_i there exists a neighbourhood U of x such that $U \cap A_i$ is disjoint from $\varphi^{-1}(\Phi_i)$. We put $F = A_0$, and for each i when A_i is already determined we put $A_{i+1} = A_i$ in case $\varphi^{-1}(\Phi_{i+1})$ dose not exist (i.e. $\varphi(F)$ is disjoint from Φ_{i+1}) and we determine in the following way in case $\varphi^{-1}(\Phi_{i+1})$ exists. Let $\{(V_\lambda, B_\lambda)\}_{\lambda \in \Lambda}$ be a maximal family of ordered pairs (V_λ, B_λ) with the following three conditions:

(1) B_λ is everywhere second category in non-void open set V_λ.

(2) $A_i \supset B_\lambda$ and B_λ is disjoint from $\varphi^{-1}(\Phi_{i+1})$.

(3) $V_\lambda \cap V_{\lambda'} = \phi$ if $\lambda \neq \lambda'$.

Put $A_{i+1} = \bigcup_{\lambda \in \Lambda} B_\lambda$. Suppose there exists an open set W such that $W \cap V_\lambda = \phi$ for all $\lambda \in \Lambda$. As Φ_{i+1} is an S-filter in E, $\varphi^{-1}(\Phi_{i+1})$ is also an S-filter in F, and hence, $\varphi^{-1}(\Phi_{i+1})$ has a countable basis $\{S_n\}$ such that $\bigcap_n S_n = \phi$. Since $W \cap A_i$ is second category and $W \cap A_i = \bigcup_n (W \cap A_i \sim S_n)$, $C = W \cap A_i \sim S_{n_0}$ is second category for some n_0. Then $W \cap A_i \supset C$ and C is disjoint from $\varphi^{-1}(\Phi_{i+1})$. Putting $V = O(C)$, and $B = V \cap C$, we obtained a pair (V, B) and the family $\{(V_\lambda, B_\lambda), (V, B)\}$ which also satisfies (1), (2) and (3), contradicting the maximality of $\{(V_\lambda, B_\lambda)\}_{\lambda \in \Lambda}$. So we have proved that $\bigcup_{\lambda \in \Lambda} V_\lambda$ is dense in F, and then it is obvious that A_{i+1} is everywhere second category.

Let D_i be the set of those $x \in F$ for which there exists a neighbourhood U of x such that $U \cap A_i$ is disjoint from $\varphi^{-1}(\Phi_i)$, and put $D = \bigcap_i D_i$. Then D_i is open and dense since $D_i \supset A_i$, and hence $F \sim D$ is first category.

Now we prove that φ is continuous on D. We apply Lemma 1 for φ, D and $D \cap A_i$, then it is sufficient to prove that $\varphi(\Phi_x)$ converges to $\varphi(x)$ for every $x \in D$, where Φ_x is defined as in the lemma. Every ultrafilter $\Psi \supset \varphi(\Phi_x)$, being disjoint from every Φ_i, converges to an element x in E. Since Φ_x converges to x and φ is graph closed, we have $z = \varphi(x)$.

Corollary. *Every graph closed linear mapping φ from a linear topological space of second category into a linear topological space E which is quasi-Souslin is continuous.*

Proof. By the theorem, there exists a subset D of F such that $F \sim D$ is first category and φ is continuous on D. Then, by Lemma 2, φ is countinuous.

References

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[2] A. Martineau: Sur le théorème du graphe fermé. C. R. Acad. Sc. Paris Série A, **263**, 870–871 (1966).