120. *Some Remarks on Boundedness of Linear Transformations from Banach Spaces into Orlicz Spaces of Lebesgue-Bochner Measurable Functions*

By Joseph DIESTEL
West Georgia College and Southwire Co.,
Carrollton, Georgia 30117, U. S. A.

(Comm. by Kinjirô KUNUGI, M. J. A., June 12, 1970)

Let X be an abstract set.

Let R be the set of real numbers. Let R^+ be the set of non-negative reals. Let Y, Z and W be arbitrary Banach spaces.

Denote by $| \quad |$ the norm of any element of Y, Z or W.

A collection V (non-empty) of subsets of X is said to be a pre-ring of subsets of X whenever $A_1, A_2 \in V$ implies $A_1 \cap A_2 \in V$ and $A_1 | A_2$ can be written as a disjoint union of some finite collection of members of V.

Let V be a pre-ring of subsets of X.

A function $v: V \to R^+$ will be called a volume whenever for every countable family of disjoint sets $A_t \in V (t \in T)$ such that $A = \bigcup_{t \in T} A_t \in V$ we have $v(A) = \sum_T v(A_t)$.

Let v be a volume defined on V. We call the triple (X, V, v) a volume space. Denote by V_v^+ the collection $\{A \in V : v(A) > 0\}$.

In [1], is developed the basic theory of the space of Lebesgue-Bochner summable functions generated by the volume space (X, V, v). We denote this space by $L_1(v, Y)$; also we denote by $S(V, Y)$ the space of all V-simple functions with values in Y, i.e., functions $f: X \to Y$ of the form $f(x) = \sum_{i=1}^{n} y_i X_{A_i}(x)$ where $y_1, \cdots, y_n \in Y$ and $A_1, \cdots, A_n$ are disjoint members of V, and by $S^+(V)$ the set of non-negative members of $S(V, R)$.

Let $f: X \to Y$. We call f v-locally summable, denoted by $f \in L_1^{loc}(v, Y)$, whenever for each $A \in V_v^+$, $X_A \cdot f \in L_1$. We endow $L_1^{loc}(v, Y)$ with the locally convex topology generated by the family of seminorms $\{\| \quad \|_A : A \in V^+\}$ where

$$\|f\|_A = \|X_A \cdot f\|_{1,v}$$

Let (p, q) be a pair of real-valued functions defined on the interval $\langle 0, \infty)$ which satisfy the following conditions: (i) p is continuous, $p: \langle 0, \infty) \to \langle 0, \infty)$, and p is differentiable with derivative p' on $(0, \infty)$; (ii) p is a diffeomorphism of $(0, \infty)$ with itself such that $q(s)$

*) The author's research on this paper was supported in part by West Georgia College Faculty Grant 699.

$=\int_0^s (p')^{-1}(t)dt$, for all $s \in [0, \infty)$.

Let $Q(V)=\left\{s \in S^+(V): \int q \circ s \, dv \leq 1\right\}$.

Let $L_p(v, Y)$ be the space of functions $f: X \to Y$ satisfying (i) $f \in L_1^{\mathrm{loc}}(v, Y)$; (ii) f has V-σ-support and (iii) $\|f\|_{p,v} < \infty$ where

$$\|f\|_{p,v}=\sup\left\{\int |f| s \, dv : s \in Q(V)\right\}$$

In [2], is developed the basic theory of the spaces $L_p(v, Y)$.

We begin with the following simple:

Theorem 1. *Let $g \in L_p(v, Y)$, and let $A \in V_v^+$. Then*

$$\int_A |g| \, dv \leq k_A \|g\|_{p,v}$$

where

$$k_A=[q^{-1}(v(A)^{-1})]^{-1}$$

Consequently, convergence in $L_p(v, Y)$ implies convergence in $L_1^{\mathrm{loc}}(v, Y)$.

Proof. Let $A \in V_v^+$. Then $s=k_A^{-1}X_A$ is clearly a member of $Q(V)$. Thus $k_A^{-1}\int_A |g| \, dv=\int s|g| \, dv \leq \|g\|_{p,v}$. The first assertion follows from this and the second assertion is just a simple consequence of the given inequality.

Theorem 1 allows us now to apply the same reasoning as in the proof of Theorem 1 in [3] yielding the following:

Theorem 2. *Let T be a linear function from Z into $L_p(v, Y)$. For each $A \in V$, define $T_A: Z \to Y$ to be the linear function*

$$T_A z=\int_A (Tz)(x) \, dv(x).$$

A necessary and sufficient condition that T be bounded is that for each $A \in V_v^+$, T_A is bounded.

Denote by $M_p(V, W)$ the space of all finitely additive functions $\mu: V \to W$ for which (i) $v(A)=0$ implies $\mu(A)=0$, and (ii) $\|\mu\|_{p,v} < \infty$ where $\|\mu\|_{p,v}=\sup\{|\sum_{i=1}^n a_i\mu(A_i)| : a_1, \cdots, a_n \in R^+, A_1, \cdots, A_n \in V$, disjoint and $\sum_1^n a_i X_{A_i} \in Q(V)\}$.

The spaces $M_p(V, W)$ are introduced in [4] where they are shown to be intimately connected with the representation of bounded linear operators from Orlicz spaces of Lebesgue-Bochner measurable functions to any Banach space.

In particular it is easily seen that for any $f \in L_p(v, W)$, then $\mu_f: V \to W$ defined by $\mu_f(A)=\int_A f \, dv$ is a member of $M_p(V, W)$ with $\|\mu_f\|_{p,v} \leq \|f\|_{p,v}$.

Using these facts it is not difficult to conclude the following extension of Theorem 2 in [3]:

Theorem 3. *Let T be a bounded linear operator from Z into $L_p(v, Y)$. Then there exists a unique $\mu_T: V \to B(Z; Y)$ (=bounded, linear operators from Z to Y) such that for each $z \in Z$, $\mu_T(\cdot)z \in M_p(V, Y)$ with*

$$(Tz)(\cdot) = \frac{d\mu_T}{dv}(\cdot)(z)(v - a.e.).$$

Thus function μ_T is given by the formula

$$(\mu_T)(A)(z) = T_A z.$$

References

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