114. *On D-dimensions of Algebraic Varieties*

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The purpose of this note is to give an outline of our recent results on D-dimensions of algebraic varieties. Details will be published elsewhere.[1]

Letting k denote an algebraically closed field of characteristic zero, we shall work in the category of schemes over k. Let V be a complete algebraic variety of dimension n, and let D be a divisor on V. We denote by $l(D)-1$ the dimension of the complete linear system $|D|$ associated with D. We consider the set of all positive integers m satisfying $l(mD)>0$, which we indicate by $N(D)$. Assume that $N(D)$ is not empty. Then $N(D)$ forms a sub-semigroup of the additive group of all integers. Hence, letting $m_0(D)$ be the g.c.d. of the integers belonging to $N(D)$, we can find a positive integer $N(D)$ such that $m \in N(D)$ if $m \equiv 0$ mod. $m_0(D)$, $m \geqq N(D)$.

Theorem 1. *There exist positive numbers α, β and a non-negative integer κ such that the following inequality holds for every sufficiently large integer m:*

$$\alpha m^{\kappa} \leqq l(mm_0(D)D) \leqq \beta m^{\kappa}.$$

It is easy to check that κ is independent of the choice of α and β. *We define the D-dimension of V to be the integer κ,* provided that $l(mD)>0$ for at least one positive integer m. We denote the D-dimension of V by $\kappa(D, V)$. In the case in which $l(mD)=0$ for every positive integer m, *we define the D-dimension of V to be* $-\infty$: $\kappa(D, V)=-\infty$.

Theorem 2. *Assume that $\kappa(D, V)>0$. For an arbitrarily fixed integer p which is greater than a constant depending on D, there exists a positive number γ such that the following inequality holds for every sufficiently large integer m:*

$$l(mm_0(D)D)-l(\{mm_0(D)-pm_0(D)\}D) \leqq \gamma m^{\kappa-1}, \quad \kappa=\kappa(D, V).$$

We recall that, in classical algebraic geometry, the index of an algebraic system on an algebraic variety of dimension n is defined to be the number of those distinct members of the system which pass through r independent generic points of V, where $r=$ the dimension of the system + the dimension of its member $-n+1$.

Theorem 3. *Suppose that $\kappa=\kappa(D, V)$ is positive. Then there*

1) On D-dimensions of algebraic varieties (to appear in Journal of Mathematical Society of Japan).

exists a κ dimensional irreducible algebraic system of algebraic varieties of dimension $n-\kappa$ with index 1, such that $\kappa(D_u, V_u)=0$, where V_u denotes a general member of the algebraic system and D_u the induced divisor on V_u of D. Moreover, such an algebraic system is unique up to birational equivalence.

Note that Theorem 3 looks like hoary but the proof of this theorem requires the new algebraic geometry developed by A. Grothendieck in E.G.A. For a better formulation of the following theorem, we introduce the notion of the co-D-dimension of V, which we write $c\kappa(D, V)$, by setting $c\kappa(D, V)=$ the dimension of V minus $\kappa(D, V)$.

Theorem 4. *Let $\tilde{V}$, V be complete algebraic varieties and let f be a proper surjective morphism from $\tilde{V}$ to V. For any divisor D on V, we have $\kappa(f^*D, \tilde{V})=\kappa(D, V)$. Moreover, if a general fiber $\tilde{V}_v=f^{-1}(v)$ is irreducible, then for any divisor $\tilde{D}$ on $\tilde{V}$, we have $c\kappa(\tilde{D}, \tilde{V})\geqq c\kappa(\tilde{D}_v, \tilde{V}_v)$.*

In order to define the canonical dimension of an arbitrary algebraic variety V, we take a non-singular projective model V^* of V, whose existence is assured by a celebrated theorem of Hironaka. Then *we define the canonical dimension $\kappa(V)$ of V to be $\kappa(K^*, V^*)$*, where K^* denotes a canonical divisor of V^*. Since the m-genus of an algebraic variety is a birational invariant for every integer $m\geqq 1$, $\kappa(V)$ is well defined and is a birational invariant. By applying the above theorems to the case in which $V=V^*$, $D=K^*$, we obtain the following theorems.

Theorem 5. *If $\kappa=\kappa(V)$ is positive, then there exists a fiber space $f: V^*\rightarrow W$ of non-singular projective algebraic varieties such that*

i) *V^* is birationally equivalent to V,*

ii) *W is of dimension κ,*

iii) *f is surjective and proper,*

iv) *any general fiber $V^*_w=f^{-1}(w)$ is irreducible and of canonical dimension 0.*

Moreover, such a fiber space is unique up to birational equivalence.

The former part of this theorem is a direct generalization of Lemma 7 in [3] which says that a surface with linear genus$=1$ and a plurigenus$\geqq 2$ is elliptic. Moreover, the latter part is a generalization of Proposition 7 in [2].

Theorem 6. *Let $\tilde{V}$, V be non-singular projective algebraic varieties and f a proper surjective morphism from $\tilde{V}$ to V. In the case in which $\tilde{V}$ is étale over V, we have $\kappa(\tilde{V})=\kappa(V)$. On the other hand, in the case in which any general fiber $f^{-1}(u)=\tilde{V}_u$ is irreducible, we have $c\kappa(\tilde{V})\geqq c\kappa(\tilde{V}_u)$.*

The former assertion is a generalization of a theorem in the theory of algebraic surfaces to the effect that every unramified covering manifold of an elliptic surface is also elliptic. The latter is a generalization of a theorem[2] saying that every algebraic surface of general type cannot contain a pencil of elliptic curves.

The canonical dimension would seem to be the most fundamental invariant in the theory of birational classification of algebraic varieties. Our theorems concerning $\kappa(D, V)$ and $\kappa(V)$ establish some fundamental results in the theory of birational classification. In particular, Theorem 5 shows that it would be enough to consider algebraic varieties of co-canonical dimension zero,[3] of canonical dimension zero and of canonical dimension $-\infty$ in order to classify algebraic varieties to the extent that Italian algebraic geometers did for algebraic surfaces about sixty years ago.

We note that the above theorems have counterparts in the category of complex spaces.[4]

References

[1] A. Grothendieck: Élements de Géométrie Algébrique. Publ. Math. de I. H. E. S. Nos. 4, 7, 11,$\cdots$.

[2] S. Iitaka: Deformations of compact complex surfaces. II. J. Math. Soc. Japan, **22**, 247-261 (1970).

[3] K. Kodaira: On the structure of compact complex analytic surfaces. I. Amer. J. Math., **86**, 751-798 (1964).

[4] L. Roth: Algebraic Threefolds. Berlin (1955).

[5] I. Šafarevič and others: Algebraic Surfaces. Proc. of Steklov Inst. of Math., Moscow (1965).

[6] O. Zariski: The theorem of Riemann-Roch for high multiples of an effective divisor on an algebraic surface. Ann. of Math., **76**, 560-615 (1962).

2) Lemma 5 in Chapter 6 in [5].

3) The co-canonical dimension of an algebraic variety V of dimension n is defined to be $n-\kappa(V)$.

4) The existence of a non-singular model of any compact complex variety was recently proved by Hironaka.