110. $\bar{l}$-Spaces over Locally Convex Spaces*)

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1. In the previous note [3], we defined the l^p-space over a Banach space and used it for a study of polynomial maps of Banach spaces. It seems to be more useful to define a similar space for a locally convex topological vector space. In this note we shall do this.

Let E be a locally convex (topological vector) space and S be its (irreducible) spectrum of seminorms [2]. Then the n^{th} tensor power $E^{\otimes n}$ of E with the projective topology admits as its spectrum the irreducible hull of the set of seminorms $\{\mathfrak{p}^{\otimes n} \mid \mathfrak{p} \in S\}$ where $\mathfrak{p}^{\otimes n}$ is a seminorm defined by $\mathfrak{p}^{\otimes n}(x) = \inf\{\sum \mathfrak{p}(x_1^{(i)}) \cdots \mathfrak{p}(x_n^{(i)}) \mid x = \sum x_1^{(i)} \otimes \cdots \otimes x_n^{(i)}\}$ for $x \in E^{\otimes n}$. For any $p, 1 \leq p < \infty$, and for any $\mathfrak{p} \in S$, we define a real valued function $l^p\mathfrak{p}$ on the (algebraic) vector space $\bigoplus_{n=1}^{\infty} E^{\otimes n}$ by $l^p\mathfrak{p}(x) = (\sum \mathfrak{p}^{\otimes n}(x_n)^p)^{1/p}$ for $x = \sum x_n, x_n \in E^{\otimes n}$. It is clearly a seminorm. Let l^pS be the irreducible hull of seminorms $\{l^p\mathfrak{p} \mid \mathfrak{p} \in S\}$, we define a locally convex space $\bar{l}^pE$ to be the set $\{x = \sum x_n \mid x_n \in E^{\otimes n}$ and $l^p\mathfrak{p}(x) < \infty$ for any $\mathfrak{p} \in S\}$ with the spectrum l^pS, and $\bar{l}^q_sE$ to be its subspace of symmetric elements. Then the following properties are easily verified.

Proposition 1. *If E is a Frechet space, so are $\bar{l}^pE$ and $\bar{l}^p_sE$. If E is Frechet and nuclear, then we have $(\bar{l}^pE)' \cong \bar{l}^qE'$ and $(\bar{l}^p_sE)' \cong \bar{l}^q_sE'$ where E' is the strong dual of E and $1/p + 1/q = 1$.*

As usual, we have $\bar{l}^pE \subset \bar{l}^qE$ if $p \leqq q$, moreover we have

Theorem 1. *For any $p, q \geq 1$, $\bar{l}^pE \subset \bar{l}^qE$ and the inclusion is continuous.*

Lemma. *For any sequence $\{a_n\}$ of positive numbers with $\lim a_n^{1/n} = 0$ and for any real $s \geqq l$, we have $(\sum a_n)^s \leqq \sum (2^n a_n)^s$.*

This Lemma is easily verified.

Proof of Theorem 1. For any $\mathfrak{p} \in S_E$ and $x_n \in E^{\otimes n}$, we have $t\mathfrak{p} \in S_E$ for any $t > 0$ and $(t\mathfrak{p})^{\otimes n}(x_n) = t^n \mathfrak{p}^{\otimes n}(x_n)$, hence $x = \sum x_n \in \bar{l}^pE$ for some p if and only if $\lim (\mathfrak{p}^{\otimes n}(x_n))^{1/n} = 0$. Then $x \in \bar{l}^qE$ for any q. This means that $\bar{l}^pE$ and $\bar{l}^qE$ coincide with each other as sets. Let $p \geqq q \geqq 1$. Let $a_n = (\mathfrak{p}^{\otimes n}(x_n))^q$ for an element $x = \sum x_n \in \bar{l}^pE$ and a seminorm $\mathfrak{p} \in S_E$, then $s = p/q \geqq 1$, $\lim a_n^{1/n} = 0$ and $\mathfrak{p}^{\otimes n}(x_n)^p = a_n^s$ hence, by the above Lemma, we have $(l^q\mathfrak{p}(x_n))^p = (\sum \mathfrak{p}^{\otimes n}(x_n)^q)^{p/q} = (\sum a_n)^s \leqq \sum (2^n a_n)^s = \sum ((2\mathfrak{p})^{\otimes n}(x_n))^p = (l^p(2\mathfrak{p})(x))^p$. Let $\mathfrak{q}$ be any seminorm in l^qS_E, then

*) Dedicated to Professor A. Komatu on his sixtieth birthday.

there is a seminorm $\mathfrak{p}_0 \in S_E$ with $\mathfrak{q} \leqq l^q \mathfrak{p}_0$. As is seen above we have $l^q \mathfrak{p}_0 \leqq l^p(2\mathfrak{p}_0)$, so that $\mathfrak{q} \leqq l^p(2\mathfrak{p}_0)$. This implies that the inclusion $\bar{l}^p E \subset \bar{l}^q E$ is continuous. q.e.d.

By the above Theorem we can identify any $\bar{l}^p E$ with each other for $1 \leqq p < \infty$. Hence we shall denote this space simply by $\bar{l}E$ if we need not refer to p. The space $\bar{l}_S E$ is defined similarly.

Next let E be a Banach space with the norm $\| \ \|$ and S_E be the spactrum consisting of seminorms $\mathfrak{p}_t = t\| \ \|$ for $t > 0$. Then it is clear that the topology defined by S_E is the underlying locally convex topology of the Banach space E. However we have

Proposition 2. *The topology of $\bar{l}^p E$ defined by $l^p S_E$ is strictly finer that the topology induced from the underlying locally convex topology of the Banach space $l^p E$.*

Proof. Since $x = \sum x_n \in \bar{l}^p E$ if and only if $(\sum t^{np} \|x^{\otimes n}\|_n^p)^{1/p} < \infty$ for any $t > 0$, it is clear that $\bar{l}^p E \subset l^p E$ and the inclusion is continuous. Choose an element $x_0 \in E$ such that $\|x_0\| = \mathfrak{p}_1(x_0) = 1/2$ and define a sequence $\{x^{(k)}\}$ in $\bar{l}^p E$ by $x^{(k)} = \sum x_n^{(k)}$ where $x_n^{(k)} = 0$ if $k \neq n$ and $x_k^{(k)} = x_0^{\otimes k}$. Then $\|x^{(k)}\| L^p = (\sum \|x_n^{(k)}\|_n^p) = 1/2^k$, hence $x^{(k)}$ converges to 0 in the topology induced from $l^p E$, but $x^{(k)}$ does not converge to 0 in the topology defined by $l^p S_E$ because $l^p \mathfrak{p}_t(x^{(k)}) = (\sum (t^n \|x_n^{(k)}\|_n)^p)^{1/p} = t^k / 2^k \to \infty$ if $t > 2$.

q.e.d.

2. Let $f: E \to F$ be a continuous linear map of locally convex spaces, then we define a linear map $lf: \bar{l}E \to \bar{l}F$ by $lf(x) = \sum f^{\otimes n}(x_n)$ for $x = \sum x_n$, $x_n \in E^{\otimes n}$. It is easily seen that $lf(\bar{l}_s E) < \bar{l}_s F$. In contrast with the case of l^p-spaces over Banach spaces [3], we have

Proposition 3. *The map $lf: \bar{l}E \to \bar{l}F$ is continuous for any continuous linear map $f: E \to F$.*

Proof. It suffices to prove that $lf: \bar{l}^p E \to \bar{l}^p F$ is continuous for some $p \geqq 1$. By definition, for any seminorm $\mathfrak{q} \in \bar{l}^p S_F$ there is a seminorm $\mathfrak{q}_0 \in S_F$ such that $\mathfrak{q} \leqq l^p \mathfrak{q}_0$. Since f is continuous, there is a seminorm $\mathfrak{p} \in S_E$ such that $\mathfrak{q}_0 \circ f \leqq \mathfrak{p}$, so we have $\mathfrak{q} \circ lf \leqq l^p \mathfrak{p}$. q.e.d.

The derivative of a map $f: E \to F$ of locally convex spaces is defined to be the map $df: E \to L(E, F)$ such that for any seminorm $\mathfrak{q} \in S_F$ there is a seminorm $\mathfrak{p} \in S_E$ with $\lim_{v \to 0} (\mathfrak{q}(f(x+v) - f(x) - df(x)(v)) / \mathfrak{p}(v) = 0$. The k^{th} derivative $d^k f: E \to L(E_s^{\otimes n}, F)$ is defined inductively by $d^k f = d(d^{k-1} f)$ and f is of class C^k if $d^k f$ is continuous.

Now, as in [3], we define a map $e: E \to \bar{l}_s E$ by $e(x) = \sum (1/n!) x^{\otimes n}$ for $x \in E$. Then easily we have

Theorem 2. *The map $e: E \to \bar{l}_s E$ is of class C^∞.*

More generally, let A be the set of non-increasing sequences $a = \{a_n\}$ of positive numbers such that $\lim a_n^{1/n} = 0$, and for any $a \in A$ we define a map $\varepsilon_a: E \to \bar{l}_s E$ by $\varepsilon_a(x) = \sum a_n x^{\otimes n}$ for $x \in E$. Then we have also

Theorem 2a. *The map $\varepsilon_a: E\to\bar{l}_sE$ is of class C^∞ for any $a\in A$.*

A map $f: E\to F$ of locally convex spaces is said to be *polynomial* (resp. *a-polynomial*, for $a\in A$) if there is a continuous linear map φ: $\bar{l}_sE\to F$ such that $f=\varphi\circ e$ (resp. $f=\varphi\circ\varepsilon_a$).

Let $P_a(E,F)$ be the vector space of a-polynomial maps from E to F. Then it is easily seen that if $a\leqq b$ (i.e. if $a_n\leqq b_n$ for each n) $P_a(E,F)\subset P_b(E,F)$ and, as in [3], we have

Theorem 3. *The map $\varepsilon_a^*: L(\bar{l}_sE,F)\to P_a(E,F)$ defined by $\varepsilon_a^*(\varphi)=\varphi\circ\varepsilon_a$ for $\varphi\in L(\bar{l}_sE,F)$ is an (algebraic) isomorphism for any $a\in A$.*

Let E, F and G be three locally convex spaces, then it follows from Proposition 3 that

Proposition 4. *$L(F,G)\circ P_a(E,F)\subset P_a(E,G)$ and $P_a(F,G)\circ L(E,F)\subset P_a(E,G)$ for any $a\in A$.*

A composition of two a-polynomial maps is not necessarily a-polynomial, but we have

Proposition 5. *For any two sequences $a, b\in A$, there is a sequence $c\in A$ such that $P_b(F,G)\circ P_a(E,F)\subset P_c(E,G)$.*

We shall call a map $f: E\to F$ an *entire map* if there is a sequence $a\in A$ such that $f\in P_a(E,F)$ and let $E(E,F)$ be the vector space of entire maps from E to F, then Proposition 5 is restated as $E(F,G)\circ E(E,F)\subset E(E,G)$. If we define a topology on $P_a(E,F)$ such that $\varepsilon_a^*: L(\bar{l}_sE,F)\to P_a(E,F)$ is a topological isomorphism (for some (fixed) topology on $L(\bar{l}_sE,F)$). Then the inclusion map $P_a(E,F)\subset P_b(E,F)$, for $a, b\in A$ with $a\leqq b$, is continuous. Thus we can define the inductive limit topology on $E(E,F)$ (cf [1]).

References

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