152. *On the Convergence Criteria of Fourier Series*

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1. Introduction and Theorems.

1.1. We consider functions f which are even, integrable on the interval $(0,\pi)$ and are periodic with period 2π. The Young-Pollard convergence criterion of Fourier series of f reads as follows [1]:

Theorem YP. *If*

(1) $$\int_0^t f(u)du=o(t) \quad as \quad t\to 0$$

and

(2) $$\int_0^t |d(uf(u))| \leqq At \quad as \quad t\to 0,$$

then the Fourier series of f converges at the origin.

This was generalized by H. Lebesgue [1] (or [2]) in the following form:

Theorem L. *If the condition* (1) *holds and*

(3) $$\int_t^\pi |f(u)-f(u+t)|u^{-1}du=o(1) \quad as \quad t\to 0,$$

then the Fourier series of f is convergent at the origin.

Later this was further generalized by S. Pollard [1]:

Theorem P. *If the condition* (1) *holds and*

(4) $$\lim_{k\to\infty} \limsup_{t\to 0} \int_{kt}^\pi |f(u)-f(u+t)|u^{-1}du=0,$$

then the Fourier series of f is convergent at the origin.

Does there exist any convergence criterion which contained in Theorem P but not in Theorem L? If there exists, Theorem P is properly more general than Theorem L. One of the object of this paper is to give an answer to this question.

1.2. On the other hand, M. and S. Izumi [3] proved the following

Theorem I. *If* (1) *holds and*

(5) $$\int_t^\pi |d(u^{-a}f(u))| \leqq At^{-a} \quad as \quad t\to 0$$

for an a, $0<a<1$, then the Fourier series of f converges at the origin.

Later B. Kwee [4] proved the

Theorem K. *The condition* (2) *implies* (5) *and the condition* (5) *implies* (4). *That is, Theorem I contains Theorem YP as a particular case, but is contained in Theorem P as a particular case.*

The question arises: what is the relation between Theorems L and

I? Answering this question, we prove the following

Theorem 1. *There is a function f which satisfies the conditions* (1) *and* (5), *but not* (3).

Since the function f satisfying the condition (5) is necessarily bounded, we see that

Theorem 1. *Theorems L and I are mutually exclusive.*

Theorem 1 gives also a solution of the problem in § 1.1.

1.3. Theorem YP is extended by G. Sunouchi [5] as follows:

Theorem S. *Let $b \geqq 1$. If*

$$\int_0^t f(u)du = o(t^b) \quad as \quad t \to 0 \tag{6}$$

and

$$\int_0^t |d(u^b f(u))| \leqq At \quad as \quad t \to 0, \tag{7}$$

then the Fourier series of f is convergent at the origin.

We prove the following theorem containing both of Theorems I and S as particular cases.

Theorem 2. *Let $b \geqq 1$. If the condition* (6) *holds and*

$$\int_{t^{1/b}}^{\delta} |d(u^{-a} f(u))| \leqq At^{-a} \quad as \quad t \to 0 \tag{8}$$

for an a, $0<a<1$ and for a δ, $0<\delta<\pi$, then the Fourier series of f is convergent at the origin.

In this direction following theorems due to J. J. Gergen [6] are known:

Theorem G. *Let $b \geqq 1$. If the condition* (6) *holds and*

$$\int_{t^{1/b}}^{\pi} |f(u)-f(u+t)| u^{-1} du = o(1) \quad as \quad t \to 0,$$

then the Fourier series of f is convergent at the origin.

Theorem G′. *Let $b \geqq 1$. If the condition* (6) *holds and*

$$\lim_{k\to\infty} \limsup_{t\to 0} \int_{(kt)^{1/b}}^{\pi} |f(u)-f(u+t)| u^{-1} du = 0,$$

then the Fourier series of f is convergent at the origin.

Evidently, Theorem G is a particular case of the Theorem G′.

Analogously to Theorem K and Theorem 1, we have

Theorem 3. *Theorem 2 is a particular case of Theorem G′ and Theorems 2 and G are mutually exclusive.*

We shall omit the proof of this theorem, since its proof is similar to those of Theorems 1 and K.

2. Proof of Theorems.

2.1. Proof of Theorem 1. Let us consider the even function f defined by

$$\begin{aligned} f(u) &= c_k \sin M_k u^k \quad \text{on } (1/n_k, 1/m_k)\ (k=2,3,\cdots) \\ &= 0, \text{ otherwise on } (0,\pi) \end{aligned} \tag{9}$$

where $c_k=1/\log k$, $m_k=2^{2^k}$, $n_k=k\cdot m_k=k\cdot 2^{2^k}$ and $M_k=\log k\cdot 2^{k\cdot 2^k}$. Evidently $f(u)\to 0$ as $u\to 0$, and then f satisfies the condition (1).

For any t, there is an integer j such that $1/m_{j+1}\leqq t<1/m_j$. If $1/n_j<t<1/m_j$, then

$$\int_t^\pi |d(u^{-a}f(u))|=\int_t^{1/m_j}+\sum_{k=1}^{j-1}\int_{1/n_k}^{1/m_j}=U_j(t)+\sum_{k=1}^{j-1}U_k$$

where

$$U_j(t)\leqq Ac_j\int_t^{1/m_j}u^{-a-1}du+c_jjM_j\int_t^{1/m_j}u^{j-a-1}du$$
$$\leqq At^{-a}+Ac_jM_j/m_j^{j-a}\leqq At^{-a}+Am_j^a\leqq At^{-a}$$

and similarly $U_k\leqq Am_k^{-a}$ and then $\sum_{k=1}^{j-1}U_k\leqq Am_{j-1}^a\leqq At^{-a}$. In the case $1/m_{j+1}\leqq t\leqq 1/n_j$, we get also the same estimation, and hence the condition (5) holds.

On the other hand,

$$\int_{1/n_k}^\pi |f(u)-f(u+1/n_k)|u^{-1}du$$
$$\geqq c_k\int_{1/n_k}^{1/m_k-1/n_k}|\sin M_ku^k-\sin M_k(u+1/n_k)^k|u^{-1}du$$
$$=\frac{c_k}{k}\int_{(1/n_k)^k}^{(1/m_k-1/n_k)^k}|\sin M_kv-\sin M_k(v^{1/k}+1/n_k)^k|v^{-1}dv$$
$$=2\frac{c_k}{k}\int_{(1/n_k)^k}^{(1/m_k-1/n_k)^k}\left|\sin\frac{M_k}{2}((v^{1/k}+1/n_k)^k-v)\cos\frac{M_k}{2}((v^{1/k}+1/n_k)^k+v)\right| v^{-1}dv.$$

The arguement of sine in the last integral changes from a small positive value to $A\log k$ and the form of the curve of the sine function in the integral is locally almost proportional to the sine curve and the similar holds for cosine function in the last integral. Thus we have

$$\int_{1/n_k}^\pi |f(u)-f(u+1/n_k)|u^{-1}du\geqq Ac_k\log\frac{n_k}{m_k}\geqq A.$$

Therefore the condition (3) is not satisfied by the function defined by (9).

2.2. Remark of Theorem 1. As a corollary of Theorem 1 we get

Corollary. *There is an even function f which satisfies the condition* (5) *but not* (3).

We can prove this more simply than Theorem 1. For, we consider the even function f defined by

$$f(u)=\sin M_ku^2 \quad \text{on } (1/n_k, 2/n_k)\ (k=1,2,\cdots)$$
$$=0, \text{ otherwise on } (0,\pi)$$

where $n_k=2^{2^k}$ and $M_k=2^{2^{k+1}}=n_k^2$ $(k=1,2,\cdots)$. Then

$$\int_{1/n_k}^\pi |f(u)-f(u+1/n_k)|u^{-1}du\geqq\int_{1/n_k}^{2/n_k}|\sin M_ku^2|u^{-1}du\geqq A\log 2$$

and the condition (5) is checked as in the proof of Theorem 1.

2.3. Proof of Theorem 2. It is sufficient to prove that

$$s_n=\int_0^\delta f(t)t^{-1}\sin nt\,dt\to 0 \quad \text{as} \quad n\to\infty$$

in the case $b>1$. We write

$$s_n=\left(\int_0^{(1/n)^{1/b}}+\int_{(1/n)^{1/b}}^\delta\right)f(t)t^{-1}\sin nt\,dt=I_1+I_2.$$

Putting $F(t)=\int_0^t f(u)\,du$ and using integration by parts, we get

$$I_1=[F(t)t^{-1}\sin nt]_0^{(1/n)^{1/b}}-\int_0^{(1/n)^{1/b}}F(t)\,t^{-2}(nt\cos nt-\sin nt)dt$$
$$=o((1/n)^{(b-1)/b})+o\left(n\int_0^{(1/n)^{1/b}}t^{b-1}dt\right)=o(1).$$

If we write $g(t)=t^{-a}f(t)$, then $g(t)=O(t^{-ab})$ by the condition (8). Since $S(t)=\int_t^\delta \frac{\sin nu}{u^{1-a}}du=O(1/nt^{1-a})$ as $t\to 0$, we get

$$I_2=[g(t)S(t)]_{(1/n)^{1/b}}^\delta-\int_{(1/n)^{1/b}}^\delta S(t)\,dg(t)$$
$$=O\left(\frac{1}{n^{(1-a)(1-1/b)}}\right)+O\left(\frac{1}{n}\int_{(1/n)^{1/b}}^\delta\frac{|dg(t)|}{t^{1-a}}\right)$$
$$=O(1/n^{(1-a)(1-1/b)})=o(1).$$

Thus the theorem is proved.

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