192. *On Some Theorems of Berberian and Sheth*

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1. Introduction. In this paper an operator T means a bounded linear operator acting on a complex Hilbert space H.

Following Halmos [4] we define the numerical range $W(T)$ as follows:

$$W(T)=\{(Tx, x)\,;\,\|x\|=1\}.$$

The basic facts concerning $W(T)$ are that it is convex and that its closure $\overline{W(T)}$ contains the spectrum $\sigma(T)$ of T.

Definition 1 ([4]). An operator T is said to be *convexoid* if

$$\overline{W(T)}=\text{co}\,\sigma(T)$$

where the bar denotes the closure and co $\sigma(T)$ means the convex hull of the spectrum $\sigma(T)$ of T.

It is known that hyponormal operator is convexoid.

S. K. Berberian introduced the notion "*cramped*" of the unitary operator as follows:

Definition 2 ([1]). An unitary operator is said to be *cramped* if its spectrum is contained in some semicircle of the unit circle

$$\{e^{i\theta}\,;\,\theta_1\leqq\theta\leqq\theta_2,\ \theta_2-\theta_1<\pi\}.$$

Definition 3. A closed sector $\mathcal{S}$ is said to be *cramped sector if*

$$\mathcal{S}=\{re^{i\theta}\,;\,r\geqq 0,\ \theta_1\leqq\theta\leqq\theta_2,\ \theta_2-\theta_1<\pi\}$$

and $\theta_2-\theta_1$ is named to be *sector angle* of cramped sector $\mathcal{S}$.

Two lines are said to be the sector lines respectively which start the origin through the end point of the semicircle of the cramped sector, that is to say, a cramped sector consists of two sector lines and a semicircle. [cf., L_1, L_2 of Fig.]

Definition 4 ([2] [7]). An operator T is said to satisfy the condition G_1 if

$$\|(T-\lambda)^{-1}\|\leqq[\text{dist}\,(\lambda, \sigma(T))]^{-1}$$

for all $\lambda\notin\sigma(T)$.

Definition 5 ([9]). A point α of $\sigma(T)$ is a *semibare point* if it lies on the circumference of some closed disk which contains no other point of $\sigma(T)$.

The set of all semibare points of $\sigma(T)$ will be denoted by $SB(\sigma(T))$. We decompose $T=UR$, polar decomposition of T. In this paper we

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shall discuss the correlation between $\overline{W(T)}$ and $\overline{W(U)}$ in view of the standpoint of the two sectors in which $\overline{W(T)}$ and $\overline{W(U)}$ lie under the appropriate conditions and we shall give the graphic representation having their geometric signification by Figure.

Our central result is as follows. If T is invertible convexoid operator which has the polar decomposition $T=UR$ with the cramped unitary operator U, then (i) $0 \notin \overline{W(T)}$, (ii) $\overline{W(T)}$ and $\overline{W(U)}$ have the same cramped sector, that is to say, they have the same sector angle and have the approximate point spectrums of T and U respectively on the same common sector lines. Moreover we shall give some theorems of hyponormal operator and operator which satisfies the condition G_1.

At the end of the introduction we should like to express here our thanks to Professor M. Nakamura for his kind suggestion.

2. Sheth proved the following theorem in [6]

Theorem A. *Let T be hyponormal and suppose that $S^{-1}TS=T^*$ where $0 \notin \overline{W(S)}$. Then T is selfadjoint.*

Here we shall show that a modification of the hypothesis of Theorem A insures the normality of T by the simple calculation.

Theorem 1. *If T is hyponormal and unitarily equivalent to its adjoint, then T is normal.*

Proof. It is sufficient to show that $\|Tx\|=\|T^*x\|$ for any vector x. Suppose that $T^*=U^*TU$ where U is unitary. Then we have

$$T=T^{**}=(U^*TU)^*=U^*T^*U$$

so that

$$\|T^*x\|\leqq\|Tx\|=\|U^*T^*Ux\|=\|T^*Ux\|\leqq\|TUx\|=\|U^*TUx\|=\|T^*x\|$$

for any vector x, that is to say, $\|Tx\|=\|T^*x\|$, or T is normal.

3. Sheth stated in [5] the following

Theorem 2. *If T satisfies the condition G_1, then the residual spectrum $R_\sigma(T)$ contains no semibare point of the spectrum $\sigma(T)$:*

$$SB(\sigma(T))\cap R_\sigma(T)=\emptyset.$$

In this section, we shall here give a proof of Theorem 2 basing on the following theorem due to Berberian [2; Lemma 2]:

Lemma 1. *If T satisfies the condition G_1 and λ is a semibare point of $\sigma(T)$, then*

$$\ker(T-\lambda)=\ker(T^*-\lambda^*),$$

where ker A is the null space of A.

Proof of Theorem 2. Let $\lambda \in SB(\sigma(T))\cap R_\sigma(T)$, then λ^* belongs to the point spectrum $P_\sigma(T^*)$ of T^*. However by Lemma 1, $\lambda \in P_\sigma(T)$, this contradiction proves Theorem 2.

4. In [1] Berberian showed the following result:

Theorem B. *If T is invertible normal operator which has the polar decomposition $T=UR$ with the cramped unitary operator U,* *then* $0 \notin \overline{W(T)}$.

He left a question for a general invertible operator. In [3] Durszt answered this question negatively by an example. However, Sheth [5] proved that the question of Berberian is still affirmative for a class of invertible hyponormal operators. In the following theorem, we shall extend the theorem of Sheth for a more wider class of convexoid operators because hyponormal operator is convexoid.

Theorem 3. *If $T=UR$ is an invertible convexoid operator such that U is cramped, then $0 \notin \overline{W(T)}$.*

To prove the theorem, we need the following well known theorem of Williams [8].

Lemma 2. *If $0 \notin \overline{W(A)}$, then $\sigma(A^{-1}B) \subset \overline{W(B)}/\overline{W(A)}$ for any operator B.*

Proof of Theorem 3. It is clear that U^{-1} is a cramped unitary operator, i.e. $0 \notin \overline{W(U^*)}$. Hence we have by Lemma 2

$$\sigma(T)=\sigma((U^{-1})^{-1}R) \subset \overline{W(R)}/\overline{W(U^*)}.$$

Since T is convexoid, we have

$$\text{conv}\,\sigma(T)=\overline{W(T)} \subset \text{conv}\,[\overline{W(R)}/\overline{W(U^*)}]. \tag{1}$$

Hence it is sufficient to prove the theorem that one has

$$0 \notin \text{conv}\,[\overline{W(R)}/\overline{W(U^*)}] \tag{2}$$

Suppose the contrary. Then there are a positive number $\varepsilon(0<\varepsilon<1)$, positive number $x, y \in \overline{W(R)}$ (because R is strictly positive) and complex $a, b \in \overline{W(U^*)}$ such that

$$\varepsilon\frac{x}{a}+(1-\varepsilon)\frac{y}{b}=0$$

Therefore we have

$$a=-\frac{\varepsilon}{1-\varepsilon}\cdot\frac{x}{y}b. \tag{3}$$

Since x, y, ε, and $1-\varepsilon$ are positive, and since the closed numerical range of an operator is convex, (3) implies $0 \in \overline{W(U^*)}$ which is a contradiction. Hence (2) is proved.

Theorem 4 ([1] [8]). *If $0 \notin \overline{W(T)}$, then U the unitary part of T is cramped, that is to say precisely, $\overline{W(U)}$ lies in the cramped sector which is enclosed by the unit circle and the two sector lines of the sector of $\overline{W(T)}$.*

Proof. Let $T=UR$ and

$$\mathcal{S}(\overline{W(T)})=\{re^{i\theta}\,;\, r>0, \theta_1 \leqq \theta \leqq \theta_2, \theta_2-\theta_1<\pi\} \tag{4}$$

where $\mathcal{S}(\overline{W(T)})$ denotes the cramped sector of $\overline{W(T)}$. Since $\overline{W(T)}$ contains the spectrum $\sigma(T)$ of T and (4) yields that T is invertible and U is unitary. By Lemma 2 and (4), U is cramped as follows

$$\sigma(U)=\sigma(TR^{-1}) \subset \overline{W(T)}/\overline{W(R)} \subset \{e^{i\theta}\,;\, \theta_1 \leqq \theta \leqq \theta_2, \theta_2-\theta_1<\pi\}$$

since U is convex,

$$\text{co}\,\sigma(U)=\overline{W(U)}\subset\{re^{i\theta}, 0<r\leqq 1\,;\, \theta_1\leqq\theta\leqq\theta_2, \theta_2-\theta_1<\pi\}$$

hence proof is complete.

Here we can precisely sharpen Theorem 3 by using the notion of sector lines, and we represent the figure having the geometric signification in which show the correlation between $\overline{W(T)}$ and $\overline{W(U)}$ under the hypothesis of Theorem 3.

Theorem 5. *If $T=UR$ is an invertible convexoid operator such that U is cramped, then*

(i) $0\notin\overline{W(T)}$

(ii) $\overline{W(T)}$ *and* $\overline{W(U)}$ *have the same cramped sector, that is to say precisely, they have the same sector angle and have the approximate point spectrums of T and U respectively on the same common sector lines.* [*Fig.*]

(iii) T_1, T_2 *in Figure are the approximate point spectrums of T and the bare points of $\overline{W(T)}$ respectively.*

Proof. We cite (1) in the proof of Theorem 3

$$\text{conv}\,\sigma(T)=\overline{W(T)}\subset\text{conv}\,[\overline{W(R)}/\overline{W(U^*)}]. \tag{5}$$

Let

$$\sigma(U)=\{e^{i\theta}, \theta_1\leqq\theta\leqq\theta_2, \theta_2-\theta_1<\pi\}$$

then

$$\overline{W(U^*)}=\text{co}\,\sigma(U^*)=\text{co}\,\{e^{-i\theta}\,;e^{i\theta}\in\sigma(U)\}$$

strict positivity of R yields $0\notin\overline{W(R)}$ and by (5) we have

$$\overline{W(T)}=\text{conv}\,\sigma(T)\subset\text{conv}\left[\frac{\overline{W(R)}}{\text{conv}\,[e^{-i\theta}\,;\,e^{i\theta}\in\sigma(U)]}\right] \tag{6}$$
$$\subset\{re^{i\theta}\,;\,r>0, \theta_1\leqq\theta\leqq\theta_2, \theta_2-\theta_1<\pi\}.$$

Hence (6) implies (i) which is already proved in Theorem 3 and by Theorem 4 and (6), we have (ii), and (iii) follows from the relation $\overline{W(T)}=\text{co}\,\sigma(T)$. $\overline{W(U)}$ lies in the sector OS_1S_2—the triangular OS_1S_2 in Figure and S_1, S_2 are the approximate point spectrums of U and the bare points of $\overline{W(U)}$ respectively.

Theorem 6. *If $T=UR$ is convexoid, then the following conditions are equivalent*

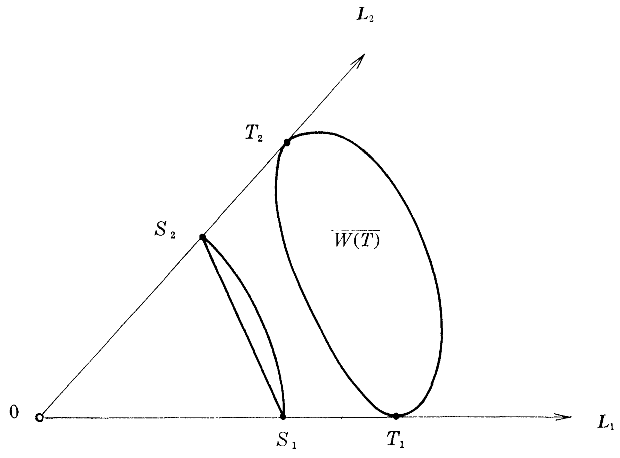
(i) *T is invertible and U is cramped*

(ii) $0\notin\overline{W(T)}$

Proof. (ii)→(i) is clear by Theorem 4 and the reverse relation is proved by Theorem 5.

Theorem 6 implies the following theorem of Berberian

Corollary 1 ([1]). *If N is normal operator, then $0\notin\overline{W(N)}$ if and only if N is invertible and $N(N^*N)^{-1/2}$ is cramped.*



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