1. *Even Maps from Spheres to Spheres*

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(Comm. by Kenjiro SHODA, M. J. A., Jan. 12, 1971)

1. Introduction. The n-sphere S^n is the set of vectors in Euclidean space R^{n+1} having unit length. An even map f from S^n to a topological space X is a continuous map preserving base points which satisfies $f(-x)=f(x)$ for any $x \in S^n$.

In this note we deal with the general problem of representing homotopy classes by even maps from spheres to spheres.

To state the results, we denote by $\widetilde{KO}^*$ the functor in the real K-theory [2]. Suppose $k\equiv 0, 1, 2$ or $4 \bmod 8$, then we have

Theorem 1.1. *An element α of the homotopy group $\pi_{n+k}(X)$ of a finite CW-complex X which induces non-zero homomorphism α^*: $\widetilde{KO}^n(X)\to\widetilde{KO}^n(S^{n+k})$ ($\approx Z$ or Z_2) can not be represented by any even map in the following cases:*

i) $n\equiv 2 \bmod 4$ *if* $k\equiv 1 \bmod 8$,

ii) $n\equiv 0$ or $3 \bmod 4$ *if* $k\equiv 2 \bmod 8$,

iii) $n\equiv 0 \bmod 2$ *if* $k\equiv 0 \bmod 4$.

By the methods of H. Toda and J. F. Adams, we have a family of the elements $\mu_{s,n}$ of $\pi_{n+k}(S^n)$ if $k=8s+1$ and $n\geq 3$. We note that $\mu_{0,n}$ is the $(n-2)$-fold suspension $\eta_n=S^{n-2}\eta_2$, where η_2 is the homotopy class of the Hopf map from S^3 to S^2.

Corollary 1.2. *Suppose $k=8s+1$ and $n\geq 3$, then*

i) *$\mu_{s,n}$ can not be represented by any even map if $n\not\equiv 2 \bmod 4$,*

ii) *$\mu_{s,n}\eta_{n+k}$ can not be represented by any even map if $n\equiv 0$ or* $3 \bmod 4$.

By Theorem 2 of [8], η_{n-1} can not be represented by any polynomial map from S^n to S^{n-1} if n is a power of 2. Since a form of even degree is an even map, Corollary 1.2 partially generalizes the above result of R. Wood.

We denote by ι_n the homotopy class of the identity of S^n and by ν_n the generator of the 2-component of $\pi_{n+3}(S^n)\approx Z_{24}$ for $n\geq 5$.

Theorem 1.3. i) *Suppose $n+k\equiv 2 \bmod 4$, then $\alpha\eta_{n+k}$ and $\alpha\eta^2_{n+k}$ are represented by even maps for any $\alpha \in \pi_{n+k}(S^n)$ respectively.*

ii) *Suppose $n+k\equiv 1 \bmod 4$ and $n\geq k+5$ and let $\alpha \in \pi_{n+k}(S^n)$ be of order 2, then we have the following.*

a) *Any element of the Toda bracket $\{\alpha, 2\iota_{n+k}, \eta_{n+k}\}$ is represented by an even map.*

b) *Assume that* $\alpha\nu_{n+k}$ *is represented by an even map, then any element of the* Toda *bracket* $\{\alpha, 2\iota_{n+k}, \eta^2_{n+k}\}$ *is represented by an even map.*

Corollary 1.4. *Suppose* $k=8s+1$, *then*

i) $\mu_{s,n}$ *and* $\mu_{s,n}\eta_{n+k}$ *are represented by even maps respectively if* $n\equiv 2 \bmod 4$ *and* $n\geq k+3$,

ii) $\mu_{s,n}\eta_{n+k}$ *is represented by an even map if* $n\equiv 1 \bmod 4$ *and* $n\geq 3$.

I do not know whether an element in the above can be represented by a quadratic form (see Corollary 1.10 of [3]).

2. Proof of Theorem 1.1. Consider the following cofibering sequence:

(2.1) $\cdots \longrightarrow S^{n+k} \xrightarrow{\pi} P(n+k) \xrightarrow{i} P(n+k+1) \xrightarrow{p} S^{n+k+1} \longrightarrow \cdots,$

where $P(n)$ *is the real projective space,* $\pi=\pi_n: S^n\to P(n)$ *the identification map,* $i=i_n: P(n)\to P(n+1)$ *the inclusion and* $p=p_{n+1}: P(n+1)\to S^{n+1}=P(n+1)/P(n)$ *the collapsing map.*

Lemma 2.1. *If* $f: S^{n+k}\to X$ *is an even map and if it induces a non-trivial homomorphism* $f^*: \widetilde{KO}^n(X)\to\widetilde{KO}^n(S^{n+k})$, *then* $i^*: \widetilde{KO}^n(P(n+k+1))\to\widetilde{KO}^n(P(n+k))$ *is not onto.*

Proof. From the first assumption there exists a map $\bar{f}: P(n+k)\to X$ such that $f=\bar{f}\pi$. By the relation $f^*=\pi^*\bar{f}^*$ and by the second assumption, π^* is non-trivial. The following exact sequence induced by (2.1) leads us to the assertion:

(2.2) $$\cdots \longleftarrow \widetilde{KO}^n(S^{n+k}) \xleftarrow{\pi^*} \widetilde{KO}^n(P(n+k)) \xleftarrow{i^*} \widetilde{KO}^n(P(n+k+1)) \xleftarrow{p^*} \widetilde{KO}^n(S^{n+k+1}) \longleftarrow \cdots.$$

The above lemma and the following one complete the proof of Theorem 1.1 in case of i) and ii).

Lemma 2.2. *Suppose* $k\equiv 1$ *or* $2 \bmod 8$, *then* $i^*: \widetilde{KO}^n(P(n+k+1))\to\widetilde{KO}^n(P(n+k))$ *is onto in the following cases:*

i) $n\not\equiv 2 \bmod 4$ *if* $k\equiv 1 \bmod 8$,

ii) $n\equiv 0$ *or* $3 \bmod 4$ *if* $k\equiv 2 \bmod 8$.

This lemma is proved by use of Theorem 1 of [4] and (2.2).

The assertion of Theorem 1.1 in case of iii) is obvious since $\widetilde{KO}^n(P(n+k))$ is finite by Theorem 1 of [4] if $n+k$ is even.

Thus the proof of Theorem 1.1 is completed.

3. A family of the elements $\mu_{s,n}$ and the proof of Corollary 1.2. We define an element $\mu'_s\in\pi_{8s+6}(S^5)$ as follows:

(3.1) $\mu'_0=\eta_5$ *and* $\mu'_s\in\{\alpha'_s, 2\iota_{8s+4}, \eta_{8s+4}\}$ *for* $s\geq 1$,

where α'_s *is the element defined at* (5.7) *of* [5].

Put

(3.2) $\alpha_{s,n}=S^{n-5}\alpha'_s$,

(3.3) $\mu_{s,n}=S^{n-5}\mu'_s$ *and* $\mu_s=S^\infty\mu'_s$.

On the other hand we can define an element $\bar{\mu}'_s \in \pi_{8s+4}(S^3)$ as follows:

(3.4) $\bar{\mu}'_0=\eta_3$, $\bar{\mu}'_1=\mu_3$ (see p. 56 of [6]), $\bar{\mu}'_2=\bar{\mu}_3$ (see p. 136 of [6]) *and* $\bar{\mu}'_s \in \{\bar{\mu}'_{s-1}, 2\iota_{8s-4}, 8\sigma_{8s-4}\}$ *for* $s\geq 3$, *where* σ_n *is the generator of the* 2*-component of* $\pi_{n+7}(S^n)\approx Z_{240}$ *for* $n\geq 9$.

Put

(3.5) $\bar{\mu}_{s,n}=S^{n-3}\bar{\mu}'_s$ *and* $\bar{\mu}_s=S^\infty\bar{\mu}'_s$.

By Proposition 3.2. (a) and 7.1 of [1], $\mu_s\equiv\bar{\mu}_s \bmod \operatorname{Ker} d_R$, where d_R is the homomorphism such that $d_R(\alpha)=\alpha^*: \widetilde{KO}^n(S^n)\to\widetilde{KO}^n(S^{n+8s+1})$ for $\alpha\in\pi_{n+8s+1}(S^n)$.

We put

(3.6) $\mu_{s,n}=\bar{\mu}_{s,n}$ *for* $n=3$ *and* 4.

Now we are ready for proving Corollary 1.2. It is obtained from Theorem 1.1 in case of i) and ii) by using the Bott periodicity theorem and Theorem 1.2 of [1].

4. Proof of Theorem 1.3. For a finite CW-complex X whose dimension is less than $2n-2$, we denote by $\pi^n(X)$ the n-th cohomotopy group of X.

From the Hopf-Eilenberg classification theorem and from Theorem 4 of [7], we have

$$(4.1)\quad \pi^n(P(n))=\{p_n\}\approx\begin{cases} Z & \text{if } n \text{ is odd,}\\ Z_2 & \text{if } n \text{ is even,}\end{cases}$$

$$(4.2)\quad p_n\pi_n=\begin{cases} 2\iota_n & \text{if } n \text{ is odd,}\\ 0 & \text{if } n \text{ is even,}\end{cases}$$

where the same symbol is used for a map and its homotopy class.

Lemma 4.1. *Let* $\bar{p}_n\in\pi^n(P(n+1))$ *and* $\overline{\eta_n p_{n+1}}\in\pi^n(P(n+2))$ *be extensions of* p_n *and* $\eta_n p_{n+1}$ *for even* $n\geq 4$ *respectively, then we have the following.*

i) *If* $n\equiv 0 \bmod 4$, *then* $2\bar{p}_n=\eta_n p_{n+1}$.

ii) *If* $n\equiv 2 \bmod 4$, *then*

a) $\bar{p}_n$ *is of order* 2 *and* $\bar{p}_n\pi_{n+1}=\eta_n$,

b) $\overline{\eta_n p_{n+1}}$ *is of order* 2 *and* $\overline{\eta_n p_{n+1}}\pi_{n+2}=\eta_n^2$.

Proof. Consider the following exact sequence:

$$(4.3)\quad \begin{aligned} &\pi^n(S^n)\xleftarrow{\pi^*}\pi^n(P(n))\xleftarrow{i^*}\pi^n(P(n+1))\xleftarrow{p^*}\pi^n(S^{n+1})\\ &\xleftarrow{S\pi^*}\pi^{n-1}(P(n))\longleftarrow\pi^{n-1}(P(n+1))\\ &\longleftarrow\pi^{n-1}(S^{n+1})\xleftarrow{S\pi^*}\pi^{n-2}(P(n))\longleftarrow\cdots.\end{aligned}$$

It follows from (4.1), (4.2), (4.3) and Corollary 1.2. i) for $k=1$ that $\pi^n(P(n+1))$ is generated by $\bar{p}_n$ and $\eta_n p_{n+1}$ and it is isomorphic to Z_4 or Z_2+Z_2 for even $n\geq 4$.

By use of (11.16) of [6], $\pi^n(P(n+1))\approx\pi^n(P(n+1)/P(n-2))\approx\pi^n(K\vee S^{n+1})$ if $n\equiv 2 \bmod 4$ and $\pi^n(P(n+1))\approx\pi^n(K\cup_{\iota_1\eta_{n-1}}e^{n+1})$ if n

$\equiv 0 \bmod 4$, where $K=S^{n-1}\cup_{\pm 2\iota_{n-1}} e^n$ and $i_1: S^{n-1}\to K$ is the inclusion.

Clearly $\pi^n(K)=\{p_1\}\approx Z_2$, where $p_1: K\to S^n$ is the collapsing map. So we have the first of ii). a).

We denote by $i_2: K\to L=K\cup_{i_1\eta_{n-1}} e^{n+1}$ the inclusion and by $p_2: L\to S^{n+1}$ the collapsing map. In the cofibering exact sequence

$$0\xleftarrow{(i_1\eta_{n-1})^*}\pi^n(K)\xleftarrow{i_2^*}\pi^n(L)\xleftarrow{p_2^*}\pi^n(S^{n+1}),$$

there exists an element $\bar{p}_1\in\pi^n(L)$ such that $\bar{p}_1 i_2=p_1$. By the definition of the Toda bracket and by Proposition 1.4, 1.3 and 1.2. i) of [6], $2S\bar{p}_1 = 2\{Sp_1, S(i_1\eta_{n-1}), p_2\} = S(\{2\iota_n, p_1, i_1\eta_{n-1}\}p_2) = S(\{2\iota_n, p_1, i_1\}\eta_n p_2) = S(\eta_n p_2)$. This leads us to i).

When $n\equiv 2 \bmod 4$, η_n can be represented by a quadratic form (see p. $140\sim 141$ of [7]) and so by an even map. From this the second of ii). a) holds.

(4.3) and ii). a) lead us to the first of ii). b).

By (4.1), (4.2), (4.3) and the second of ii). a), we have

(4.4) $\pi^n(P(n+1))=0$ *if* $n\equiv 3 \bmod 4$.

By i) $S(\eta_n p_{n+1})=2S\bar{p}_n=2\{Sp_n, S\pi_n, p_{n+1}\}=S(\{2\iota_n, p_n, \pi_n\}p_{n+1})$. So we have, by (4.4),

(4.5) $\{2\iota_n, p_n, \pi_n\}=\eta_n$ *if* $n\equiv 0 \bmod 4$.

By Proposition 1.2. iii) and 1.4 of [6] and by (4.2), (4.4) and (4.5), $S(\overline{\eta_n p}_{n+1}\pi_{n+2})=\{S(\eta_n p_{n+1}), S\pi_{n+1}, p_{n+2}\}S\pi_{n+2}=\{\eta_{n+1}, 2\iota_{n+2}, p_{n+2}\}S\pi_{n+2} =\eta_{n+1}\{2\iota_{n+2}, p_{n+2}, \pi_{n+2}\}=\eta_{n+1}^2$ for $n\equiv 2 \bmod 4$. This leads us to the second of ii). b).

Thus the proof of the lemma is completed.

Now we shall prove Theorem 1.3.

i) of Theorem 1.3 is a direct consequence of ii) of Lemma 4.1.

By Proposition 1.3 and 1.2. i) of [6] and by ii). a) of Lemma 4.1, $S\{\alpha, 2\iota_{n+k}, \eta_{n+k}\}=\{S\alpha, 2\iota_{n+k+1}, \eta_{n+k+1}\}=\{S\alpha, 2\iota_{n+k+1}, \bar{p}_{n+k+1}\pi_{n+k+2}\}=\{S\alpha, 2\iota_{n+k+1}, \bar{p}_{n+k+1}\}S\pi_{n+k+2}$. This is contained in $\pi^{n+1}(SP(n+k+2))S\pi_{n+k+2} =S(\pi^n(P(n+k+2))\pi_{n+k+2})$ since $n\geq k+5$. So we have ii). a) of Theorem 1.3.

The proof of ii). b) of Theorem 1.3 is similar to the above and we omit it.

5. Proof of Corollary 1.4. ii) of the corollary is obvious.

We put $k=8s-1$ and $\alpha=\alpha_{s,n}$ in Theorem 1.3. ii). If $n\geq k+5 =(8s+1)+3$, then we have, by (3.1) and (3.3), $\mu_{s,n}\in\{\alpha_{s,n}, 2\iota_{n+k}, \eta_{n+k}\}$ and $\mu_{s,n}\eta_{n+k+2}\in\{\alpha_{s,n}, 2\iota_{n+k}, \eta_{n+k}^2\}$ since $\alpha_{s,n}\nu_{n+k}=0$ by Lemma 5.1. i) of [5]. Consequently i) of Corollary 1.4 holds.

By use of Theorem 1.3 we have the following for sufficiently large n.

Example 5.1. i) *If* $k=8s-1$ *and* $n\equiv 3 \bmod 4$, *then* $\rho_{s,n}\eta_{n+k}$ *and* $\rho_{s,n}\eta_{n+k}^2$ *which are the generators of the J-images* (see (5.17) of [5]) *are represented by even maps respectively* (see Corollary 1.10 of [3]).

ii) *If* $n\equiv 3 \bmod 4$, *then* $\varepsilon_n \in \{\nu_n^2, 2\iota_{n+6}, \eta_{n+6}\}$, $\eta_n^* \in \{\sigma_n^2, 2\iota_{n+14}, \eta_{n+14}\}$ *and* $\eta_n^*\eta_{n+16} \in \{\sigma_n^2, 2\iota_{n+14}, \eta_{n+14}^2\}$ (see ii) of Theorem 2.1 of [5]) *are represented by even maps respectively.*

References

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