31. Note on Betti-Numbers of the Module of Differentials

By Tadayuki MATSUOKA
Ehime University

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Let R be a reduced locality over a perfect field k and let $D_k(R)$ be the module of k-differentials of R (Kähler differentials of R over k). In this note we shall discuss the relations between the Betti-numbers of $D_k(R)$ and some of the algebraic or homological invariants of the k-algebra R.

Throughout this note we assume that all rings are commutative noetherian rings with identity and all modules are finitely generated and unitary.

§1. In this section we shall state notations, definitions and a preliminary lemma. Let A be a ring and M an A-module. We denote by $\dim(A)$ the Krull dimension of A, by $\mathrm{Ass}(M)$ the set of associated prime ideals of M and by $\mathrm{hd}(M)$ the homological dimension of M. In case when A is a local ring with maximal ideal $\mathfrak{m}$, we denote by $r(M)$ the number of minimal generators of M. The number $r(\mathfrak{m})$ is called the embedding dimension of A and denoted by $\mathrm{emdim}(A)$. The dimension of $\mathrm{Tor}_i^A(M, A/\mathfrak{m})$ as a vector space over $A/\mathfrak{m}$ is called the ith Betti-number of M and denoted by $b_i(M)$.

For a local ring A, a composite concept (B, f) of a regular local ring B and a surjective homomorphism $f: B \to A$ is called an embedding of A. An embedding (B, f) of A is said to be minimal if the kernel of f is contained in the square of the maximal ideal of B. It follows from the definition that if (B, f) is a minimal embedding of A, then $\dim(B) = \mathrm{emdim}(A)$.

Let A be a ring and M an A-module. We say that M is torsion free if non zero elements in M are not annihilated by non zero-divisors in A. We shall use later the following:

Lemma. *Let A be a ring and M a torsion free A-module. If $M_\mathfrak{p} = 0$ for all $\mathfrak{p}$ in $\mathrm{Ass}(A)$, then $M = 0$.*

§2. Let R be a locality over a perfect field k, i.e., R is a quotient ring of a finitely generated k-algebra with respect to a prime ideal. Let $D_k(R)$ be the module of k-differentials of R. Let $\mathfrak{m}$ be the maximal ideal of R. We denote by $\mathrm{tr.\,d.}_k(R/\mathfrak{m})$ the transendence degree of the field $R/\mathfrak{m}$ over k. Then we have the following equality:

(1) $$r(D_k(R_\mathfrak{p})) = \mathrm{emdim}(R_\mathfrak{p}) + \dim(R/\mathfrak{p}) + \mathrm{tr.d.}_k(R/\mathfrak{m})$$

for every prime ideal $\mathfrak{p}$ in R. As a special case of (1), we have

(2) $$r(D_k(R)) = \text{emdim}\,(R) + \text{tr.d.}_k(R/\mathfrak{m}).$$

From now on we assume that R is reduced, i.e., R has no non-zero nilpotent elements. From (1) it follows that

(3) $$r(D_k(R_\mathfrak{p})) \leq \dim\,(R) + \text{tr.d.}_k(R/\mathfrak{m})$$

for all $\mathfrak{p}$ in Ass (R) and the equality holds for some $\mathfrak{p}$ in Ass (R).

Let

(*) $$\cdots \to L_i \to \cdots \to L_0 \to D_k(R) \to 0$$

be a minimal free resolution of $D_k(R)$, i.e., an exact sequence of free R-modules L_i with Im $(L_{i+1} \to L_i) \subseteq \mathfrak{m}L_i$. Then $r(L_i) = b_i(D_k(R))$ and, in particular, $r(L_0) = r(D_k(R))$. Set Im $(L_{i+1} \to L_i) = N_i$. For $\mathfrak{p}$ in Ass (R) we have an exact sequence of vector spaces over the field $R_\mathfrak{p}$:

$$0 \to (N_n)_\mathfrak{p} \to (L_n)_\mathfrak{p} \to \cdots \to (L_0)_\mathfrak{p} \to D_k(R_\mathfrak{p}) \to 0.$$

Hence we have the equality

(4) $$\sum_{i=1}^{n} (-1)^{i-1} b_i(D_k(R)) = r(D_k(R)) - r(D_k(R_\mathfrak{p})) + (-1)^{n-1} r((N_n)_\mathfrak{p})$$

for every $\mathfrak{p}$ in Ass (R).

Theorem 1. *Let R be a reduced locality over a perfect field k and let $D_k(R)$ be the module of k-differentials of R. Then for every positive odd integer n the following inequality holds:*

$$\sum_{i=1}^{n} (-1)^{i-1} b_i(D_k(R)) \geq \text{emdim}\,(R) - \dim\,(R).$$

Moreover, for some odd integer n, the equality holds if and only if $\text{hd}(D_k(R)) \leq n$.

Proof. By (2), (3) and (4) we have

$$\sum_{i=1}^{n} (-1)^{i-1} b_i(D_k(R)) \geq \text{emdim}\,(R) - \dim\,(R) + r((N_n)_\mathfrak{p})$$

for every $\mathfrak{p}$ in Ass (R). From this the first statement follows. Assume that the left hand side of the above inequality is equal to emdim (R)-dim (R). Then $(N_n)_\mathfrak{p} = 0$ for all $\mathfrak{p}$ in Ass (R). Since N_n is a submodule of the free module L_n, N_n is torsion free. Hence, by Lemma in § 1, $N_n = 0$. This shows that $\text{hd}(D_k(R)) \leq n$. Conversely assume that $\text{hd}(D_k(R)) \leq n$. Since the resolution (*) is minimal, we have $N_n = 0$. Therefore, by (3) and (4), we have the required equality. q.e.d.

Remark 1. In the if part of Theorem 1, the integer n is not necessary odd.

Corollary 1 ([3]). *With the same notation and assumptions as in Theorem* 1, *the following inequality holds:*

$$b_1(D_k(R)) \geq \text{emdim}\,(R) - \dim\,(R).$$

Moreover, the equality holds if and only if $\text{ha}(D_k(R)) \leq 1$.

Corollary 2. *With the same notation and assumptions as in Theorem* 1, *if* $\text{hd}(D_k(R))$ *is finite, then the zero ideal of R is equidimensional,* i.e., $\dim\,(R/\mathfrak{p}) = \dim\,(R)$ *for all* $\mathfrak{p}$ *in* Ass (R). *Moreover, the following equality holds:*

$$\dim (R)=\mathrm{emdim}\,(R)+\sum_{i\geq 1}(-1)^i\, b_i(D_k(R)).$$

Proof. If $\mathrm{hd}(D_k(R))\leq n$, then $N_n=0$ and hence, by (4), we have $\dim (R/\mathfrak{p})=\mathrm{emdim}\,(R)+\sum_{i=1}^{n}(-1)^i b_i(D_k(R))$ for all $\mathfrak{p}$ in $\mathrm{Ass}\,(R)$. This implies that $\dim (R/\mathfrak{p})$ does not depend on $\mathfrak{p}$ in $\mathrm{Ass}\,(R)$. Since $\dim (R/\mathfrak{p})=\dim (R)$ for some $\mathfrak{p}$ in $\mathrm{Ass}\,(R)$, the assertions are proved.

§3. Let R be a reduced locality over a perfect field k and $\mathfrak{m}$ its maximal ideal. In this section we shall consider the relation between the 1st Betti-number $b_1(D_k(R))$ and the 2nd deviation $\delta_2(R)$ of R, i.e., the dimension of André's homology group $H_2(R, R/\mathfrak{m}, R/\mathfrak{m})$ as a vector space over $R/\mathfrak{m}$. (For the definition of André's homology groups see [1].)

First we state the following proposition which is a special case of Proposition in Vasconcelos [7].

Proposition. *Let A be a local ring and $\mathfrak{a}$ an ideal which has height r and finite homological dimension. If a homomorphism g of the $(A/\mathfrak{a})$-module $\mathfrak{a}/\mathfrak{a}^2$ into a free $(A/\mathfrak{a})$-module of rank r is surjective, then g is injective.*

We return to our subject. Since R is a lacality over k, R has a minimal embedding (S, f) such that S is also a locality over k. Set $\mathrm{Ker}\,(f)=\mathfrak{a}$. Then $R=S/\mathfrak{a}$ and $\dim (S)=\mathrm{emdim}\,(R)$. Hence the module $D_k(S)$ of k-differentials of S is a free S-module of rank $r(D_k(R))$. Consider the exact sequence

$$(**)\qquad \mathfrak{a}/\mathfrak{a}^2 \xrightarrow{\rho} R\otimes_S D_k(S)\rightarrow D_k(R)\rightarrow 0$$

(cf. [4]) and set $\mathrm{Im}\,(\rho)=N$. Then we have the exact sequence

$$0\rightarrow \mathrm{Tor}_1^R(D_k(R), R/\mathfrak{m})\rightarrow N/\mathfrak{m}N\rightarrow R\otimes_S D_k(S)/\mathfrak{m}(R\otimes_S D_k(S))\rightarrow D_k(R)/\mathfrak{m}D_k(R)\rightarrow 0.$$

Since $r(D_k(R))=r(R\otimes_S D_k(S))$, we have $r(N)=b_1(D_k(R))$. On the other hand, by Proposition 27.1 and Proposition 25.3 in [1], André's 2nd homology group $H_2(R, R/\mathfrak{m}, R/\mathfrak{m})$ is isomorphic to $\mathfrak{a}/\mathfrak{n}\mathfrak{a}$, where $\mathfrak{n}$ is the maximal ideal of S. Hence $\delta_2(R)=r(\mathfrak{a})$. Since obviously $r(\mathfrak{a})\geq r(N)$, by Corollary 1 to Theorem 1, we have the following inequalities (cf. [3], [6]):

$$(5)\qquad \delta_2(R)\geq b_1(D_k(R))\geq \mathrm{emdim}\,(R)-\dim (R).$$

In (5) if $b_1(D_k(R))=\mathrm{emdim}\,(R)-\dim (R)$, then $\delta_2(R)=b_1(D_k(R))$. In fact, consider the image N of the map ρ in the exact sequence (**). Since $r(N)=b_1(D_k(R))$ and since the height of $\mathfrak{a}$ is equal to $\mathrm{emdim}\,(R)-\dim (R)$, $r(N)$ is equal to the height of $\mathfrak{a}$. On the other hand by Corollary 1 to Theorem 1 N is a free R-module since $R\otimes_S D_k(S)$ is a free R-module. Hence by the alove proposition the map ρ is injective. This shows that $\delta_2(R)=b_1(D_k(R))$.

Notice that R is a complete intersection if and only if $\delta_2(R) = \text{emdim}(R) - \dim(R)$. From these and from Corollary 1 to Theorem 1 we have the following theorem in which the equivalence (i) and (ii) was given by Ferrand [2] and Vasconcelos [7].

Theorem 2. *With the same notation and assumptions as in Theorem 1, the following conditions are equivalent:*

(i) *R is a complete intersection*

(ii) $\text{hd}(D_k(R)) \leq 1$, *or equivalently* $b_2(D_k(R)) = 0$

(iii) $b_1(D_k(R)) = \text{emdim}(R) - \dim(R)$.

Corollary. *With the same notation and assumptions as in Theorem 1, if $b_1(D_k(R)) \leq 1$, then R is a complete intersection.*

The following example, due to Scheja and Storch [6], shows that in Theorem 2 the assumption that R is a reduced ring is necessary even if the ground field k is of characteristic zero, and that without this assumption the implications (iii)$\Rightarrow$(i) and (iii)$\Rightarrow$(ii) may fail to hold.

Example. Let k be a field of characteristic zero and set $S = k[X, Y]_{(X,Y)}$ and $F = X^3 + XY^5 + Y^7$. Let $\mathfrak{a}$ be the ideal generated by F and the partial derivatives F_X and F_Y. Set $R = S/\mathfrak{a}$. Then $\dim(R) = 0$ and R is not reduced. It is easy to see that $\delta_2(R) = 3$, $b_1(D_k(R)) = 2$ and $\text{emdim}(R) - \dim(R) = 2$. Since $D_k(R)$ is not free, $\text{hd}(D_k(R))$ is not finite. Therefore the equality $b_1(D_k(R)) = \text{emdim}(R) - \dim(R)$ does imply neither the equality $\delta_2(R) = b_1(D_k(R))$ nor the finiteness of $\text{hd}(D_k(R))$.

Let A be a local ring. M. André introduced in [1] a new notion "*simplicial dimension of A*" (notation: s-dim(A)) as follows:

We say that s-dim$(A) \leq n$ if $\delta_i(A) = 0$ for $i \geq n$. In our case when R is the locality considered as above, we have the following equivalent conditions:

(i) R is a regular local ring

(ii) $\text{hd}(D_k(R)) = 0$, i.e., $b_1(D_k(R)) = 0$

(iii) s-dim$(R) \leq 2$, i.e., $\delta_2(R) = 0$

and also have the equivalent conditions:

(i) R is a complete intersection

(ii) $\text{hd}(D_k(R)) \leq 1$, i.e., $b_2(D_k(R)) = 0$

(iii) s-dim$(R) \leq 3$, i.e., $\delta_3(R) = 0$

(cf. [1]).

He also presented the following questions:

(a) Does $\delta_i(A)$ vanish for large i?

(b) If s-dim(A) is finite, then is $\sum (-1)^{i+1}\delta_i(A)$ equal to $\dim(A)$?

In these questions, if we replace $\delta_{i+1}(R)$ by $b_i(D_k(R))$ for $i \geq 1$ and s-dim(R) by $\text{hd}(D_k(R))$, then the former is negative because there exist reduced localities over k such that $\text{hd}(D_k(R)) = \infty$, i.e., $b_i(D_k(R)) \neq 0$ for all $i \geq 0$ (e.g. R is a non complete intersection of Krull dimension one). However, since $\delta_1(R)=\mathrm{emdim}\,(R)$, Corollary 2 to Theorem 1 shows that the latter is affirmative.

Remark 2. As is showm in [1], for $i=1,2$ the deviations $\delta_{i+1}(R)$ are equal to the "Abweichungen $\varepsilon_i(R)$" introduced by Scheja [5].

Added in proof. In the analytic case, the equidimensionality of the zero ideal of R in Corollary 2 to Theorem 1 is also given in the lecture note by G. Scheja at the University of Genova, Differential Modules of Analytic Rings (1968).

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