86. On a Certain Difference Scheme for Some Semilinear Diffusion System

By Masayasu MIMURA, Yoshinori KAMETAKA, and Masaya YAMAGUTI

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Recently many mathematical models have been formulated by many physical mathematicians and mathematical physicists. Most of them are derived from the systems in physics, chemistry, medicine and ecology, for example, super conductivity system by T. Tsuneto and E. Abrahams [1], animal nerve axon system by A. L. Hodgkin and A. F. Huxley [2] and population dynamics system by Volterra and Kerner [3], etc. All of them are expressed by the following semilinear diffusion system:

(1) $$D_t U = A\Delta U + F(U),$$

where $U={}^t(u_1, u_2, \cdots, u_N)$, $F(U)={}^t(f_1(u_1, u_2, \cdots, u_N), f_2(u_1, u_2, \cdots, u_N), \cdots, f_N(u_1, u_2, \cdots, u_N))$, and A is a N-th order constant diagonal matrix with its non-negative element a_i for $i=1, 2, \cdots, N$.

Here we deal with the system (1) as an initial value problem in $H=\{x \in R_n, 0 \leqq t < +\infty\}$ with the initial data

(2) $$U(0, x)=\Phi(x),$$

where $\Phi={}^t(\phi_1, \phi_2, \cdots, \phi_N)$. Our concern is to formulate the stable difference scheme to resolve the problem (1) and (2).

First, we give the essential hypothesis to (1),

Hypothesis. (i) As for the matrix A,

$$a_{q_{m+1}} = a_{q_{m+2}} = \cdots = a_{q_{m+1}} = \bar{a}_{m+1},$$

where $m=0, 1, 2, \cdots, M-1$, $q_0=0$ and $q_M=N$.

(ii) For some constant vectors

$$D_m^p = (0, 0, \cdots, 0, d^p_{q_{m-1}+1}, d^p_{q_{m-1}+2}, \cdots, d^p_{q_m}, 0, \cdots, 0),$$

there exist non-negative constant c_m^p and non-negative $s_m^p(U)$ such that

$$D_m^p F(U) \leqq (c_m^p - D_m^p U) s_m^p(U)$$

for

$$U \in \bigcap_{m=1}^{M} \bigcap_{p=1}^{P(m)} \{U \in R^N, D_m^p U \leq c_m^p\},$$

where $m=1, 2, \cdots, M$ and $p=1, 2, \cdots, P(m)$.

We consider the following difference scheme to (1) and (2) under the Hypothesis,

(3) $$\frac{1}{\Delta t}(U^{i+1,J} - U^{i,J}) = \frac{1}{(\Delta x)^2} A \sum_{k=1}^{n} T_-^k T_+^k U^{i,J} + F(U^{i,J}) - S(U^{i,J})(U^{i+1,J} - U^{i,J})$$

(4) $$U^{o,J}=\Phi(J\Delta x).$$

Here $U^{i,J}=U(i\Delta t, j_1\Delta x, j_2\Delta x, \cdots, j_n\Delta x)$ for n-tuple of integers $(j_1, j_2, \cdots, j_n)$ and for a non-negative integer i, Δt and Δx are the mesh sizes of t and x directions respectively, $T^k_{\pm}$ is an operator replacing j_k by $j_k\pm 1$, i.e., $T^k_{\pm}U^{i,J}=U(i\Delta t, j_1\Delta x, j_2\Delta x, \cdots, j_{k-1}\Delta x, (j_k\pm 1)\Delta x, j_{k+1}\Delta x, \cdots, j_n\Delta x)-U^{i,J}$ and

$$S(U)=\begin{bmatrix} S_1(U) & & & \\ & S_2(U) & & O \\ O & & \ddots & \\ & & & S_M(U) \end{bmatrix}$$ and $S_m(U)$ is a q_m-th order diagonal matrix with its same elements $\sum_{p=1}^{P(m)} s^p_m(U)$.

Then we have,

Theorem. *Consider the difference scheme* (3) *and* (4) *to* (1) *and* (2). *If the set*

$$\bigcap_{m=1}^{M}\bigcap_{p=1}^{P(m)}\{\Phi\in R^N\,;\,D^p_m\Phi\leqq C^p_m\} \text{ is compact,}$$

then it follows for any i and J,

$$U^{i,J}\in\bigcap_{m=1}^{M}\bigcap_{p=1}^{P(m)}\{U^{i,J}\in R^N,\ D^p_mU^{i,J}\leqq C_m\},$$

where $0<\dfrac{\Delta t}{(\Delta x)^2}\max(\bar a_1, \bar a_2, \cdots, \bar a_M)\leqq\dfrac{1}{2n}$.

Proof. It is sufficient to prove $D^{p_0}_{m_0}U^{i,J}\leqq c^{p_0}_{m_0}$ for any m_0 $(1\leq m_0\leq M)$ and any $p_0(1\leq p_0\leq P(m_0))$. From (3), it follows that

(5) $$U^{i+1,J}=P_n(U^{i,J})+\Delta tF(U^{i,J})-\Delta tS(U^{i,J})(U^{i+1,J}-U^{i,J})$$

where $P_n(U^{i,J})=\left(I+\dfrac{\Delta t}{(\Delta x)^2}A\sum_{k=1}^{n}T^k_-T^k_+\right)U^{i,J}$. Multiplying (5) by $D^{p_0}_{m_0}$, we have

$$D^{p_0}_{m_0}\cdot U^{i+1,J}=P_n(D^{p_0}_{m_0}U^{i,J})+\Delta tD^{p_0}_{m_0}\cdot F(U^{i,J})-\Delta tD^{p_0}_{m_0}\cdot S(U^{i,J})(U^{i+1,J}-U^{i,J})$$

and because of (i) of Hypothesis,

$$\begin{aligned} D^{p_0}_{m_0}U^{i+1,J} &\leqq P_n(D^{p_0}_{m_0}U^{i,J})+\Delta t(c^{p_0}_{m_0}-D^{p_0}_{m_0}U^{i,J})s^{p_0}_{m_0}(U^{i,J}) \\ &\quad -\Delta tD^{p_0}_{m_0}S(U^{i,J})(U^{i+1,j}-U^{i,J}) \\ &=P_n(D^{p_0}_{m_0}U^{i,J})+\Delta t(c^{p_0}_{m_0}-D^{p_0}_{m_0}U^{i+1,J})s^{p_0}_{m_0}(U^{i,J}) \\ &\quad -\Delta t\sum_{\substack{p=1\\p\neq p_0}}^{P(m_0)}s^p_{m_0}(U^{i,J})(D^{p_0}_{m_0}U^{i+1,J}-D^{p_0}_{m_0}U^{i,J}). \end{aligned}$$

Thus it results that

(6) $$D^{p_0}_{m_0}U^{i+1,J}\leqq\frac{P_n(D^{p_0}_{m_0}U^{i,J})+\Delta tc^{p_0}_{m_0}s^{p_0}_{m_0}(U^{i,J})+\Delta t\sum_{\substack{p=1\\p\neq p_0}}^{P(m_0)}s^p_{m_0}(U^{i,J})D^{p_0}_{m_0}U^{i,J}}{1+\Delta t\sum_{p=1}^{P(m_0)}s^p_{m_0}(U^{i,J})}.$$

Supposing $D^p_mU^{i,J}\leqq c^p_m$ for any p and m, (6) implies from $s^p_m(U^{i,J})\geqq 0$,

$$D^{p_0}_{m_0}U^{i+1,J}\leqq c^{p_0}_{m_0}$$

Therefore, because of the arbitrariness of p_0 and m_0 and the assumption of Φ, we obtain

$$D_m^p U^{i+1,J} \leqq c_m^p$$

from any $m(1 \leq m \leq M)$ and any $p(1 \leq p \leq P(m))$. This completes the proof of the theorem.

Application. Here we apply our difference scheme (3) and (4) to three primary physical systems.

(1) Population dynamics system ($N=1$)

$$D_t u = \Delta u + (1-u)u$$
$$u(0,x) = \phi(x) \quad \text{with} \quad 0 \leq \phi(x) \leq 1.$$

Then

$$u^{i+1,J} = P_3(u^{i,J}) + \Delta t(1-u^{i+1,J})u^{i,J}.$$

(2) Super conductivity system ($N=2$)

$$D_t U = \Delta U + F(U)$$
$$U(0,x) = \Phi(x) \quad \text{with} \quad -1 \leqq \Phi(x) \leqq 1,$$

where

$$f_1(U) = (1-u_2^2-u_1^2)u_1$$
$$f_2(U) = (1-u_2^2-u_1^2)u_2.$$

Then

$$U^{i+1,J} = P_3(U^{i,J}) + \Delta t F(U^{i,J}) - 2\Delta t(u_2^2+u_1^2)^{i,J} \times (U^{i+1,J} - U^{i,J}).$$

(3) Antigen and antibody reaction system ($N=4$)

$$D_t U = A\Delta U + F(U)$$
$$U(0,x) = \Phi(x) \quad \text{with} \quad \Phi(x) \geqq 0,$$

where

$$\left.\begin{array}{l} f_1(U) = -d_1u_1u_4 - d_2u_1u_3 \\ f_2(U) = -d_3u_2u_4 + d_2u_1u_3 \\ f_3(U) = -d_2u_1u_3 + d_3u_2u_4 \\ f_4(U) = -d_1u_1u_4 - d_3u_2u_4 \end{array}\right\}, \qquad A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and d_i is all positive constant for $i=1,2,3$.

Then

$$U^{i+1,j} = P_1(U^{i,j}) + \Delta t F(U^{i,j}) - \Delta t S(U^{i,j})(U^{i+1,j} - U^{i,j}),$$

where $S(U^{i,j})$ is a fourth diagonal matrix with its elements

$$s_1(U^{i,j}) = s_2(U^{i,j}) = ((d_1+d_3)u_4 + d_2u_3)^{i,j}$$
$$s_3(U^{i,j}) = s_4(U^{i,j}) = ((d_1+d_2)u_1 + d_3u_2)^{i,j}.$$

Remark. Our scheme can be applied to the Dirichlet problem of the semilinear elliptic system which satisfies the Hypothesis.

References

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