167. On the Generalized Decomposition Numbers of the Alternating Group

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The generalized decomposition numbers of the symmetric group are rational integers ([5], [13]), but those of the alternating group are not necessarily rational integers ([5]). The main purpose of this paper is to give a proof of the following theorem ([4]).

Theorem 1. *The generalized decomposition numbers of the alternating group for $p=2$ are rational integers.*

Throughout this paper, we consider the representations of groups over the algebraically closed field of characteristic 2. Let x be a 2-element of the alternating group A_n, and let $N_A(x)$ be the normalizer of x in A_n. In section 2 we shall prove that every 2-block B_σ^* of $N_A(x)$ is characterized by a 2-core $[\alpha_0]$, and then B_σ^* with the 2-core $[\alpha_0]$ determines the 2-block B_σ of A_n with the same 2-core $[\alpha_0]$.

1. The generalized symmetric group $S(a_i, 2^i)$ is the semi-direct product of the normal subgroup Q_i of order $(2^i)^{a_i}$ and the subgroup $S_{a_i}^*$ which is isomorphic to the symmetric group S_{a_i}([9]):

(1.1) $$S(a_i, 2^i)=S_{a_i}^*Q_i, \quad S_{a_i}^*\cap Q_i=1, \quad S_{a_i}^*\cong S_{a_i}.$$

Evidently we have $S(a_0, 1)=S_{a_0}$. Since $S(a_i, 2^i)/Q_i\cong S_{a_i}$, we see that every modular irreducible character of $S(a_i, 2^i)$ is given by the modular irreducible character of S_{a_i}.

Let G be a subgroup of the symmetric group S_n and let us denote by G^+ the subgroup $G\cap A_n$ of G. Then we have $G=G^+$ or $(G: G^+)=2$. Since $(Q_i: Q_i^+)=2$ for $i>0$, we see that

(1.2) $$S(a_i, 2^i)^+=S_{a_i}^*Q_i^+.$$

Let y be an arbitrary 2-regular element of $S(a_i, 2^i)$. Then y is the even permutation and hence $y\in S(a_i, 2^i)^+$. It follows from $S(a_i, 2^i)^+/Q_i^+ \cong S_{a_i}$ that every representation of $S(a_i, 2^i)^+$ obtained by restricting the modular irreducible representation of $S(a_i, 2^i)$ remains irreducible. If we denote by φ_κ^i $(\kappa=1, 2, \cdots, m_i)$ the modular irreducible characters of S_{a_i}, then the modular irreducible characters of $S(a_i, 2^i)$ and $S(a_i, 2^i)^+$ are also given by $\varphi_\kappa^i(y)$. This implies that the representation $\tilde{U}_\kappa^i$ of $S(a_i, 2^i)$ induced from the indecomposable constituent U_κ^i of the regular representation of $S(a_i, 2^i)^+$ is the indecomposable constituent of the regular representation of $S(a_i, 2^i)$ ([8]) and hence if we denote by $\tilde{c}_{\kappa\lambda}$ and

$c_{\kappa\lambda}$ the Cartan invariants of $S(a_i, 2^i)$ and $S(a_i, 2^i)^+$ respectively, then

(1.3) $$\tilde{c}_{\kappa\lambda}=2c_{\kappa\lambda}.$$

It follows from (1.3) that two characters $\varphi^i_\kappa(y)$ and $\varphi^i_\lambda(y)$ of $S(a_i, 2^i)^+$ belong to the same 2-block, if and only if $\varphi^i_\kappa(y)$ and $\varphi^i_\lambda(y)$ of $S(a_i, 2^i)$ belong to the same 2-block.

Let $x\neq 1$ be a 2-element of A_n which consists of a_i cycles of length 2^i $(0\leqq i\leqq l, a_i\geqq 0)$. Denote by $N(x)$ the normalizer of x in S_n. Then $N_A(x)=N(x)\cap A_n$ is the normalizer of x in A_n. We have ([13])

(1.4) $$N(x)=S(a_0, 1)\times S(a_1, 2)\times \cdots \times S(a_l, 2^l)$$

and every modular irreducible character φ^x of $N(x)$ is the product of the modular irreducible characters φ^i of S_{a_i}:

(1.5) $$\varphi^x=\varphi^0\varphi^1\cdots\varphi^l.$$

If $a_0=0$, then

$$N(x)\supset N_A(x)\supseteqq S(a_1, 2)^+\times S(a_2, 2^2)^+\times \cdots \times S(a_l, 2^l)^+.$$

Hence we see easily that every modular irreducible character of $N_A(x)$ is given by

(1.6) $$\varphi^x=\varphi^1\varphi^2\cdots\varphi^l$$

If $a_0\neq 0$, then $N(x)=S_{a_0}\times T$ where

$$T=S(a_1, 2)\times S(a_2, 2^2)\times \cdots \times S(a_l, 2^l)$$

and we have

$$N_A(x)=A_{a_0}\times T^+ +(A_{a_0}\times T^+)st$$

where s and t denote the odd permutations of S_{a_0} and T respectively. We see from $st\in N_A(x)$ that two 2-regular elements which are conjugate in S_{a_0} are also conjugate in $N_A(x)$. This implies that every representation of $N_A(x)$ obtained by restricting the modular irreducible representation of S_{a_0} remains irreducible. Hence we see that every modular irreducible character of $N_A(x)$ is given by (1.5). Consequently, for $0\leqq a_0<n$ every modular irreducible character of $N_A(x)$ is given by (1.5) and the matrix Φ^x of the modular irreducible characters of $N_A(x)$ is the Kronecker product of the matrices Φ_{a_i} of the modular irreducible characters φ^i of S_{a_i}:

(1.7) $$\Phi^x=\Phi_{a_0}\times\Phi_{a_1}\times\cdots\times\Phi_{a_l}.$$

Let y be a 2-regular element of A_n such that $xy=yx$. Then we have the following lemma (cf. [13]).

Lemma 1. *Let $x\neq 1$ be a 2-element of A_n. Then the modular irreducible characters $\varphi^x(y)$ of $N_A(x)$ are rational integers.*

Now we shall give the proof of Theorem 1. Let $y_0=1, y_1, \cdots, y_{r-1}$ be a complete system of representatives for the 2-regular elements in $N_A(x)$ such that they all lie in different classes of $N_A(x)$ but that every 2-regular element in $N_A(x)$ is conjugate to one of them. Then the xy_j $(j=0, 1, \cdots, r-1)$ constitute a complete system of representatives for the classes of A_n which contain an element whose 2-factor is conjugate to x in A_n. We denote by $\zeta_0=1, \zeta_1, \cdots, \zeta_{s-1}$ the irreducible characters of A_n in the field of complex numbers and set

(1.8) $$Z^x=(\zeta_i(xy_j)).$$

If $x\neq 1$, then the $\zeta_i(xy_j)$ are rational integers. It follows from

(1.9) $$Z^x=D^x\Phi^x$$

where $D^x=(d_{i\kappa}^x)$ is the matrix of the generalized decomposition numbers $d_{i\kappa}^x$ of A_n that

(1.10) $$D^x=Z^x(\Phi^x)^{-1}.$$

This, combined with Lemma 1, yields that the $d_{i\kappa}^x$ are rational numbers. Since the $d_{i\kappa}^x$ are algebraic integers, we see that the $d_{i\kappa}^x$ are rational integers.

As an example we shall calculate the $d_{i\kappa}^x$ of A_7 for $p=2$ and $x=(45)(67)$ (cf. [5]). Since

$$N_A((45)(67))=S(3,1)^+\times S(2,2)^+ + (S(3,1)^+\times S(2,2)^+)(12)(45),$$

we have by (1.7)

$$\Phi^x=\Phi_3\times\Phi_2=\begin{bmatrix}1 & 1\\ 2 & 1\end{bmatrix}\times[1]=\begin{bmatrix}1 & 1\\ 2 & 1\end{bmatrix}.$$

Since $y_0=1, y_1=(123)$, we obtain

$$Z^x=\begin{bmatrix}1 & 1\\ 2 & 2\\ -1 & -1\\ -1 & -1\\ 1 & 1\\ 2 & -1\\ 2 & -1\\ -2 & 1\\ -2 & 1\end{bmatrix}, \quad \text{and hence} \quad D^x=\begin{bmatrix}1 & 0\\ 2 & 0\\ -1 & 0\\ -1 & 0\\ 1 & 0\\ 0 & 1\\ 0 & 1\\ 0 & -1\\ 0 & -1\end{bmatrix}.$$

2. Let U_κ^x be an indecomposable constituent of the regular representation of $N_A(x)$ and let $\tilde{U}_\kappa^x$ be the representation of $N(x)$ induced from U_κ^x. Then we see that $\tilde{U}_\kappa^x$ is the indecomposable constituent of the regular representation of $N(x)$. Let us denote by $\tilde{c}_{\kappa\lambda}$ and $c_{\kappa\lambda}$ the Cartan invariants of $N(x)$ and $N_A(x)$ respectively. We then obtain

(2.1) $$\tilde{c}_{\kappa\lambda}=2c_{\kappa\lambda}.$$

We have by (2.1) the following

Lemma 2. *Two characters φ_κ^x and φ_λ^x of $N_A(x)$ belong to the same 2-block, if and only if φ_κ^x and φ_λ^x of $N(x)$ belong to the same 2-block.*

Lemma 3. *Let*

$$\varphi_\kappa^x=\varphi_{\kappa_0}^0\varphi_{\kappa_1}^1\cdots\varphi_{\kappa_l}^l,$$
$$\varphi_\lambda^x=\varphi_{\lambda_0}^0\varphi_{\lambda_1}^1\cdots\varphi_{\lambda_l}^l$$

be two modular irreducible characters of $N_A(x)$. Then φ_κ^x and φ_λ^x belong to the same 2-block, if and only if $\varphi_{\kappa_0}^0$ and $\varphi_{\lambda_0}^0$ of S_{a_0} belong to the same 2-block.

Proof. For $i>0$, $S(a_i, 2^i)$ has only one block ([12], Lemma 10). Combining this with Lemma 2, we obtain the proof of Lemma 3.

Let us denote by B_σ^0 the 2-block of S_{a_0} which contains the character $\varphi_{\kappa 0}^0$ and by $[\alpha_0]$ the 2-core of B_σ^0. By Lemma 3, we may call $[\alpha_0]$ the 2-core of the 2-block B_σ^x which contains the character φ_κ^x.

Lemma 2, combined with [(13), Theorem 2], gives the following

Theorem 2. *Let* $[\alpha_0]$ *be the* 2*-core of the* 2*-block* B_σ^x *of* $N_A(x)$. *Then* B_σ^x *determines the* 2*-block* B_σ *of* A_n *with the same* 2*-core* $[\alpha_0]$.

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