30. A Representation of Entropy Preserving Isomorphisms between Lattices of Finite Partitions*)

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1. Introduction. We showed in [2] that the entropy in the information theory can be characterized as the semivaluation on the semilattice, and we discussed about measure preserving transformations as entropy preserving lattice-isomorphisms on the space of all measurable finite partitions. In this paper, we shall analyse the relation between measure preserving transformations and entropy preserving lattice-isomorphisms more minutely. Considering an arbitrary entropy preserving lattice-isomorphism which is defined abstractly as a mapping from the family of all finite partitions of a probability measure space onto that of another probability space, we shall see that such a lattice-isomorphism induces an isometrical isomorphism from the measure algebra of the former space onto the algebra of the latter. Hence, on some natural measure spaces, entropy preserving lattice-isomorphisms are represented as measure preserving point transformations. And we shall see that two concepts of conjugacy (cf. Billingsley [1], p. 66) and isomorphism of AD-systems (cf. [2]) are equivalent for the general dynamical systems.

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2. Notations and definitions. In what follows we deal with two probability measure spaces $(X,\mathcal{X},p)$ and $(Y,\mathcal{Y},q)$. When we indicate either $(X,\mathcal{X},p)$ or $(Y,\mathcal{Y},q)$, we represent it commonly by $(Z,\mathcal{Z},r)$. The quotient algebra $\mathcal{Z}/\mathcal{N}$, where $\mathcal{N}$ is the ideal of null sets, is simply written by $\tilde{\mathcal{Z}}$, and called a measure algebra with the measure r. The sets in $\mathcal{Z}$ are denoted by $A, B, C, \cdots$ and the elements in $\tilde{\mathcal{Z}}$ are denoted by $\tilde{A}, \tilde{B}, \tilde{C}, \cdots$, where $\tilde{A}$ represents the residue class containning the set A in $\mathcal{Z}$. The class of all finite measurable partitions of Z, which is a lattice with the order of refinement $\prec$ (and $\vee$, $\wedge$ for the two operations of join and meet for the lattice), is denoted by F_Z, and $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}, \cdots$ are elements in F_Z. The entropy function $H(\cdot)$ on F_Z is defined by

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$$H(\mathcal{A}) = -\sum_{A \in \mathcal{A}} r(A) \log r(A) \qquad (\mathcal{A} \in F_Z). \tag{2.1}$$

The entropy function is the semivaluation on the lattice F_Z, i.e., the following relation is satisfied: for $\mathcal{A}, \mathcal{B}, \mathcal{C} \in F_Z$ with $\mathcal{B} < \mathcal{C}$,

$$H(\mathcal{A} \vee \mathcal{C}) + H(\mathcal{B}) \leq H(\mathcal{A} \vee \mathcal{B}) + H(\mathcal{C}). \tag{2.2}$$

We can induce a pseudo-metric ρ in F_Z by the formula: $\rho(\mathcal{A}, \mathcal{B}) = 2H(\mathcal{A} \vee \mathcal{B}) - H(\mathcal{A}) - H(\mathcal{B})$(cf. [2]). Then we can make the quotient metric space $(\tilde{F}_Z, \tilde{\rho})$ of (F_Z, ρ). An element $\tilde{\mathcal{A}}$ ($\mathcal{A} = \{A_1, \cdots, A_n\}$) in the quotient metric space $\tilde{F}_Z$ is expressed by a family $\{\tilde{A}_1, \cdots, \tilde{A}_n\}$ of elements $\tilde{A}_i$ in $\tilde{\mathcal{Z}}$ with $\tilde{A}_i \wedge \tilde{A}_j = \tilde{\phi}_Z (i \neq j)$ and $\vee_{i=1}^n \tilde{A}_i = \tilde{Z}$. So we can write $\tilde{\mathcal{A}} = \{\tilde{A}_1, \cdots, \tilde{A}_n\}$ and see that $\tilde{F}_Z$ is also a lattice. By $n(\tilde{\mathcal{A}})$ we denote the number n of such elements $\tilde{A}_i$ with $r(\tilde{A}_i) > 0$ in $\tilde{\mathcal{A}}$. We shall use the same symbol H for the entropy on $\tilde{F}_Z$, i.e., $H(\tilde{\mathcal{A}}) = H(\mathcal{A})$ for $\mathcal{A} \in F_Z$, then H is a semivaluation on the lattice $\tilde{F}_Z$.

3. Main theorems. Let φ be a lattice-isomorphism from $\tilde{F}_X$ onto $\tilde{F}_Y$. Then $n(\tilde{\mathcal{A}}) = n(\varphi\tilde{\mathcal{A}})$ for all $\tilde{\mathcal{A}} \in \tilde{F}_X$. In fact, writing $\tilde{\mathcal{A}} = \{\tilde{A}_1, \cdots, \tilde{A}_n\}$ where $n = n(\tilde{\mathcal{A}})$, and putting $\tilde{\mathcal{A}}_1 = \{\tilde{X}\}$ and

$$\tilde{\mathcal{A}}_i = \{\tilde{A}_1, \cdots, \tilde{A}_{i-1}, \bigvee_{k=i}^{n} \tilde{A}_k\} \qquad i = 2, \cdots, n,$$

we get

$$\tilde{\mathcal{A}}_1 \prec \tilde{\mathcal{A}}_2 \prec \cdots \prec \tilde{\mathcal{A}}_n = \tilde{\mathcal{A}},$$

and $n(\tilde{\mathcal{A}}_i) = i$. As φ is a lattice isomorphism,

$$\varphi(\tilde{\mathcal{A}}_1) \prec \varphi(\tilde{\mathcal{A}}_2) \prec \cdots \prec \varphi(\tilde{\mathcal{A}}_n) = \varphi(\tilde{\mathcal{A}}),$$

$$\varphi(\tilde{\mathcal{A}}_i) \neq \varphi(\tilde{\mathcal{A}}_{i+1}) \text{ and } n(\varphi(\tilde{\mathcal{A}}_i)) < n(\varphi(\tilde{\mathcal{A}}_{i+1})) \qquad i = 1 \cdots, n-1.$$

Hence we have $n(\tilde{\mathcal{A}}) \leq n(\varphi(\tilde{\mathcal{A}}))$, and the inverse inequality is clear, as φ^{-1} is also a lattice-isomorphism.

If X is a two-point-set, and the probability measure on it is $p = (1/2, 1/2)$, then we call it a trivial case.

Theorem 1. *For any entropy preserving lattice-isomorphism φ from $\tilde{F}_X$ onto $\tilde{F}_Y$, there exists an isometrical isomorphism M from the measure algebra $\tilde{\mathfrak{X}}$ onto $\tilde{\mathfrak{Y}}$, with $\varphi\tilde{\mathcal{A}} = M\tilde{\mathcal{A}} = \{M\tilde{A}_1, \cdots, M\tilde{A}_n\}$ for any $\tilde{\mathcal{A}} = \{\tilde{A}_1, \cdots, \tilde{A}_n\} \in \tilde{F}_X$.*

Proof. First, let us show the following facts: there exists a mapping M from $\tilde{\mathfrak{X}}$ onto $\tilde{\mathfrak{Y}}$ satisfying 1° $\varphi\tilde{\mathcal{A}} = \{M\tilde{A}_1, M\tilde{A}_2\}$ for $\tilde{\mathcal{A}} = \{\tilde{A}_1, \tilde{A}_2\} \in \tilde{F}_X$, 2° $M(\tilde{A}^c) = (M\tilde{A})^c$, for any $\tilde{A} \in \tilde{\mathfrak{X}}$, 3° $\tilde{A}_1 \leq \tilde{A}_2$ $(\tilde{A}_1, \tilde{A}_2 \in \tilde{\mathfrak{X}}) \Rightarrow M\tilde{A}_1 \leq M\tilde{A}_2$, 4° $\tilde{B}_1 \leq \tilde{B}_2 (\tilde{B}_1, \tilde{B}_2 \in \tilde{\mathfrak{Y}}) \Rightarrow M^{-1}\tilde{B}_1 \leq M^{-1}\tilde{B}_2$, and 5° $p(\tilde{A}) = q(M\tilde{A})$ for any $\tilde{A} \in \tilde{\mathfrak{X}}$.

For $\tilde{A} \in \tilde{\mathfrak{X}}$ with $0 < p(A) < 1/2$, putting $\tilde{\mathcal{A}} = \{\tilde{A}, \tilde{A}^c\}$, $\varphi\tilde{\mathcal{A}}$ is two atomic and can be written by $\varphi\tilde{\mathcal{A}} = \{\tilde{B}, \tilde{B}^c\}$ for some $\tilde{B} \in \tilde{\mathfrak{Y}}$. Then we get $p(\tilde{A}) = q(\tilde{B})$ or $p(\tilde{A}) = q(\tilde{B}^c)$, because the isomorphism is entropy preserving. Assuming the former case, we define the mapping M by $M\tilde{A} = \tilde{B}$, and assuming the latter, $M\tilde{A} = \tilde{B}^c$. We can also define the mapping M for $\tilde{A} \in \tilde{\mathfrak{X}}$ with $1/2 < p(A) < 1$ by $M\tilde{A} = \{M(\tilde{A}^c)\}^c$, as

$0<p(A^c)<1/2$. Setting $M\tilde{X}=\tilde{Y}$ and $M\tilde{\phi}_X=\tilde{\phi}_Y$ (ϕ_Z is the empty set in Z), we have defined the mapping M from

$$\tilde{\mathscr{D}}_X=\{\tilde{A}\in\tilde{\mathscr{X}}: p(\tilde{A})\neq 1/2\} \text{ onto } \tilde{\mathscr{D}}_Y=\{\tilde{B}\in\tilde{\mathscr{Y}}: q(\tilde{B})\neq 1/2\}.$$

The conditions 1°, 2° and 5° are clear on $\tilde{\mathscr{D}}_X$ from the definition of M.

Now let us prove 3° on $\tilde{\mathscr{D}}_X$: let $\tilde{A}_1\leq\tilde{A}_2$ ($\tilde{A}_1\neq\tilde{A}_2$) with $p(\tilde{A}_2)<1/2$. If $q(M\tilde{A}_1\backslash M\tilde{A}_2)>0$, then $q(M\tilde{A}_1\wedge M\tilde{A}_2)>0$. Because if $q(M\tilde{A}_1\wedge M\tilde{A}_2)=0$, then putting $a=q(M\tilde{A}_1)$, $b=q(M\tilde{A}_2)$, and putting $\tilde{\mathscr{A}}_1=\{\tilde{A}_1,\tilde{A}_1^c\}$ and $\tilde{\mathscr{A}}_2=\{\tilde{A}_2,\tilde{A}_2^c\}$, we get

$$H(\tilde{\mathscr{A}}_1\vee\tilde{\mathscr{A}}_2)=-a\log a-(b-a)\log(b-a)-(1-b)\log(1-b),$$

and

$$H(\varphi\tilde{\mathscr{A}}_1\vee\varphi\tilde{\mathscr{A}}_2)=-a\log a-b\log b-(1-b-a)\log(1-b-a).$$

As the function

$$f(x)=-(b-a)\log(b-a)-(1-b)\log(1-b)+b\log b \\ +(1-b-x)\log(1-b-x)$$

is negative valued for $x<b<1$, we see

$$H(\tilde{\mathscr{A}}_1\vee\tilde{\mathscr{A}}_2)<H(\varphi\tilde{\mathscr{A}}_1\vee\varphi\tilde{\mathscr{A}}_2)=H(\varphi(\tilde{\mathscr{A}}_1\vee\tilde{\mathscr{A}}_2)),$$

which contradicts the fact that φ is entropy preserving. Hence $q(M\tilde{A}_1\wedge M\tilde{A}_2)>0$, and then $n(\tilde{\mathscr{A}}_1\vee\tilde{\mathscr{A}}_2)=3$ and $n(\varphi\tilde{\mathscr{A}}_1\vee\varphi\tilde{\mathscr{A}}_2)=4$, which is also a contradiction. Therefore $M\tilde{A}_1\leq M\tilde{A}_2$. If $\tilde{A}_1\leq\tilde{A}_2$ with $0<p(\tilde{A}_1)<1/2$ and $1/2<p(\tilde{A}_2)$, then we can also conclude that $M\tilde{A}_1\leq M\tilde{A}_2$ by the similar method of the former case. If $\tilde{A}_1\leq\tilde{A}_2$ with $1/2<p(\tilde{A}_1)$, then considering the complements of $\tilde{A}_1$ and $\tilde{A}_2$, we also have $M\tilde{A}_1\leq M\tilde{A}_2$. The condition 4° for $\tilde{B}_1\leq\tilde{B}_2$ with $\tilde{B}_1,\tilde{B}_2\in\tilde{\mathscr{D}}_Y$ follows similarly replacing φ by φ^{-1} and taking the mapping N from $\tilde{\mathscr{D}}_Y$ onto $\tilde{\mathscr{D}}_X$ instead of M and from the fact $N=M^{-1}$.

Now let us extend M to the outside of $\tilde{\mathscr{D}}_X$. For $\tilde{A}\in\tilde{\mathscr{X}}\backslash\tilde{\mathscr{D}}_X$ (i.e., $p(\tilde{A})=1/2$), if there exists $\tilde{A}'\in\tilde{\mathscr{D}}_X$ with $\tilde{\phi}_X\neq\tilde{A}'\leq\tilde{A}$, then writting $\tilde{\mathscr{A}}=\{\tilde{A},\tilde{A}^c\}$, $\tilde{\mathscr{A}}'=\{\tilde{A}',\tilde{A}'^c\}$ and $\varphi\tilde{\mathscr{A}}=\{\tilde{B},\tilde{B}^c\}$, we get $M\tilde{A}'\leq\tilde{B}$ or $M\tilde{A}'\leq\tilde{B}^c$, because $n(\tilde{\mathscr{A}}\vee\tilde{\mathscr{A}}')=n(\varphi\tilde{\mathscr{A}}\vee\varphi\tilde{\mathscr{A}}')$. If $M\tilde{A}'\leq\tilde{B}$, then we set $M\tilde{A}=\tilde{B}$, and $M\tilde{A}=\tilde{B}^c$ if $M\tilde{A}'\leq\tilde{B}^c$. This is well defined. Because if $M\tilde{A}'\leq\tilde{B}$ and another $\tilde{A}''\in\tilde{\mathscr{D}}_X$ with $\tilde{\phi}_X\neq\tilde{A}''\leq\tilde{A}$ and $M\tilde{A}''\leq\tilde{B}^c$ exists, then we can see $\tilde{A}'\vee\tilde{A}''=\tilde{A}$ and $\tilde{A}'\wedge\tilde{A}''=\tilde{\phi}_X$. As $\tilde{A}'\vee\tilde{A}''\neq\tilde{A}$ implies $\tilde{A}'\vee\tilde{A}''\in\tilde{\mathscr{D}}_X$ and $M\tilde{A}'$, $M\tilde{A}''\leq M(\tilde{A}'\vee\tilde{A}'')\leq\tilde{B}$ or $\tilde{B}^c$, which is a contradiction. And as $\tilde{A}'\wedge\tilde{A}''\neq\tilde{\phi}_X$ implies $M(\tilde{A}'\wedge\tilde{A}'')\leq M(\tilde{A}')\leq\tilde{B}$ and $M(\tilde{A}'\wedge\tilde{A}'')\leq M(\tilde{A}'')\leq\tilde{B}^c$, which is a contradiction. So $\tilde{A}'\vee\tilde{A}''=\tilde{A}$ and $\tilde{A}'\wedge\tilde{A}''=\tilde{\phi}_X$, which however contradict to the fact that φ preserves the number of atoms. Therefore M is well defined for such $\tilde{A}\in\tilde{\mathscr{X}}\backslash\tilde{\mathscr{D}}_X$.

For $\tilde{A}\in\tilde{\mathscr{X}}\backslash\tilde{\mathscr{D}}_X$ which has no non-empty subsets belonging to $\tilde{\mathscr{D}}_X$ we can assume that $\tilde{A}^c$ has some non-empty subset in $\tilde{\mathscr{D}}_X$, because: if $\tilde{A}^c$ is also an atom, then $\tilde{X}$ is two-atomic and for such spaces Theorem trivially consists without the uniqueness of M. Hence we define $M\tilde{A}$ by $(M\tilde{A}^c)^c$ in this case.

In the next stage, let us show that the extended mapping M satisfies the conditions $1^\circ \sim 5^\circ$ on all over $\tilde{\mathfrak{X}}$. The conditions 1°, 2° and 5° are clear. 3°: If $\tilde{A}_1 \leq \tilde{A}_2$ with $\tilde{A}_1 \in \tilde{\mathfrak{D}}_X$ and $\tilde{A}_2 \in \tilde{\mathfrak{X}} \setminus \tilde{\mathfrak{D}}_X$ then $M\tilde{A}_1 \leq M\tilde{A}_2$ is clear, and if $\tilde{A}_1 \in \tilde{\mathfrak{X}} \setminus \tilde{\mathfrak{D}}_X$ and $\tilde{A}_2 \in \tilde{\mathfrak{D}}_X$, then $\tilde{A}_2^c \leq \tilde{A}_1^c$ and $M\tilde{A}_2^c \leq M\tilde{A}_1^c$ implies $M\tilde{A}_1 \leq M\tilde{A}_2$. 4°: For φ^{-1} we extend the mapping N to $\tilde{\mathfrak{Y}} \setminus \tilde{\mathfrak{D}}_Y$ and we also see that $N = M^{-1}$.

Now every $\tilde{\mathcal{A}} \in \tilde{F}_X$ is the least upper bound of some two-atomic partitions $\tilde{\mathcal{A}}_1, \cdots, \tilde{\mathcal{A}}_n$, and so,

$$\begin{aligned}\varphi\tilde{\mathcal{A}} &= \varphi(\tilde{\mathcal{A}}_1 \vee \cdots \vee \tilde{\mathcal{A}}_n) = (\varphi\tilde{\mathcal{A}}_1) \vee \cdots \vee (\varphi\tilde{\mathcal{A}}_n) \\ &= (M\tilde{\mathcal{A}}_1) \vee \cdots \vee (M\tilde{\mathcal{A}}_n) = M(\tilde{\mathcal{A}}_1 \vee \cdots \vee \tilde{\mathcal{A}}_n) = M\tilde{\mathcal{A}}.\end{aligned}$$

For the proof of uniqueness of M, if φ induces another such mapping M', then $M\tilde{A} = M'\tilde{A}$ is clear for $\tilde{A} \in \tilde{\mathfrak{D}}_X$, and for $\tilde{A} \in \tilde{\mathfrak{X}} \setminus \tilde{\mathfrak{D}}_X$, which is also clear from the definition of M, excepting the trivial case of two atomic space. Q.E.D.

For two dynamical systems $(X, \mathfrak{X}, p, S)$ and $(Y, \mathfrak{Y}, q, T)$, where S and T are invertible measure preserving transformations on X and Y respectively, they are *conjugate* (cf. [1]) iff there exists an isometrical isomorphism M from $\tilde{\mathfrak{X}}$ onto $\tilde{\mathfrak{Y}}$ and $MS\tilde{A} = TM\tilde{A}$ for every $\tilde{A} \in \mathfrak{X}$. The transformations S and T can be recognized as an entropy preserving lattice-automorphisms on $\tilde{F}_X$ and $\tilde{F}_Y$ respectively. Then the triplets $(\tilde{F}_X, H, S)$ and $(\tilde{F}_Y, H, T)$, where H is the entropy function, are said to be *isomorphic as AD-systems* iff there exists an entropy preserving lattice isomorphism φ from $\tilde{F}_X$ onto $\tilde{F}_Y$, and $\varphi S\tilde{\mathcal{A}} = T\varphi\tilde{\mathcal{A}}$ for every $\tilde{\mathcal{A}} \in \tilde{F}_X$. We get:

Theorem 2. *The systems $(\tilde{F}_X, H, S)$ and $(\tilde{F}_Y, H, T)$ are isomophic as AD-systems if and only if $(X, \mathfrak{X}, p, S)$ and $(Y, \mathfrak{Y}, q, T)$ are conjugate, excepting the trivial case.*

Proof. The "if" part is clear. We assume that there exists an entropy preserving lattice isomorphism φ from $\tilde{F}_X$ onto $\tilde{F}_Y$, and $\varphi S\tilde{\mathcal{A}} = T\varphi\tilde{\mathcal{A}}$ for every $\tilde{\mathcal{A}} \in \tilde{F}_X$. Then Theorem 1 tells us that there exists a mapping M induced by φ, and $MS\tilde{\mathcal{A}} = TM\tilde{\mathcal{A}}$ $(\tilde{\mathcal{A}} \in \tilde{F}_X)$. Now $\psi = MS = TM$ is again an entropy preserving lattice isomorphism from $\tilde{F}_X$ onto $\tilde{F}_Y$, and both MS and TM are the versions of ψ as the induced mappings from $\tilde{\mathfrak{X}}$ onto $\tilde{\mathfrak{Y}}$. Hence uniqueness of the induced mapping shows that $MS\tilde{A} = TM\tilde{A}$ for every $\tilde{A} \in \tilde{\mathfrak{X}}$. Q.E.D.

Corollary. *If $(X, \mathfrak{X}, p)$ and $(Y, \mathfrak{Y}, q)$ are the abstract Lebesgue spaces, then the dynamical systems $(X, \mathfrak{X}, p, S)$ and $(Y, \mathfrak{Y}, q, T)$ are isomorphic iff $(\tilde{F}_X, H, S)$ and $(\tilde{F}_Y, H, T)$ are isomorphic as AD-systems, excepting the trivial case.*

References

[1] P. Billingsley: Ergodic Theory and Information. John Wiley and Sons, Inc. (1965).

[2] Y. Nakamura: Entropy and semivaluations on semilattices. Kōdai Math. Sem. Rep., 22, 443–468 (1970).