28. On Closed Graph Theorem. II

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This paper is to give, succeeding the investigation in the previous paper [2], another type of closed graph theorem generalizing and simplifying the result obtained in [1].

A linear topological space E is called a *G-space* if there exist countable S-filters Φ_n $(n=1,2,\cdots)$(i.e. each Φ_n has a countable basis $\{S_k\}$ such that $\bigcap_{k=1}^{\infty} S_k=\phi$) satisfying the following condition $(*)$.

$(*)$ *For any filter Ψ in E which is disjoint from every Φ_n $(n=1, 2,\cdots)$, there exist a complete metric group G and a continuous homomorphism f from G into E such that for any neighbourhood U of 0 in E, $f(U)$ absorbs*[1] *some element B in Ψ.* In the sequel, we call G-system the set of countable S-filters Φ_n $(n=1,2,\cdots)$ satisfying the condition $(*)$.

In the definition above, we can make, without altering the meaning of definition, further restrictions: (1) G is abelian and (2) f is surjective. For (2), if f is not surjective, we can replace G by $G\times E$ (giving discrete topology on E) and f by f' defined as $f'(x,y)=f(x)+y$ for $x\in G$ and $y\in E$. In the sequel we always suppose G to be abelian.

We can see easily that the class of G-spaces, as in the case of GN-spaces (in [2]), is closed under the following operations:

(1) *The image $F=\varphi(E)$ of a G-space E by a continuous linear mapping φ is a G-space.*

(2) *The sequentially closed subspace F of a G-space E is a G-space.*

(3) *The product space $E=\prod_n E_n$ of G-space E_n $(n=1,2,\cdots)$ is a G-space.*

(4) *The inductive limit E of G-spaces E_n $(n=1,2,\cdots)$ is a G-space.*

First we prove that every complete metric linear space E is a G-space. Let U be the unit ball in E and Φ be the filter generated by $E\backslash nU$ $(n=1,2,\cdots)$. Then E is a G-space with G-system $\Phi_n=\Phi$ $(n=1,2,\cdots)$.

Corresponding to the closed graph theorem for GN-spaces in [2],

1) A set A is said to absorb a set B, if there exists a positive real number α such that $\beta B\subset A$ for all β in $(0,\alpha]$.

we obtain the following

Theorem. *Every linear mapping φ with sequentially closed graph from an $\mathcal{F}$-space F into a G-space E is continuous.*

Proof. Since the graph $G(\varphi)$ of φ is a sequentially closed subspace of the product space $E\times F$ of G-spaces E and F, it is a G-space. Therefore it is sufficient to prove that every continuous linear mapping φ from a G-space E onto a $\mathcal{F}$-space F is open.

Let Φ_n $(n=1,2,\cdots)$ be a G-system in E. Then there exists a sequence of subset B_k $(k=1,2,\cdots)$ in E such that $B_k\supset B_{k+1}$, B_k is disjoint from Φ_k, and $\varphi(B_k)$ is of second category in F. Let Ψ be the filter generated by $\{B_k\,;\,k=1,2,\cdots\}$. Then Ψ is disjoint from every Φ_n $(n=1,2,\cdots)$, so there exist a complete metric group G and continuous homomorphism f from G to E such that for every neighbourhood U of 0 in G, $f(U)$ absorbs some B_k so that $\varphi\circ f(U)$ is of second category in F. For any neighbourhood V of 0 in E, since there exists a neighbourhood U of 0 in G such that $f(U)\subset V$, $\varphi(V)$ is of second category in F, and hence the closure of $\varphi(V)$ has an interior point. Now the proof is completed by virtue of the following well known fact.

Let φ be a continuous homomorphism from a complete metric group X to a metric group Y. If for every neighbourhood U of 0 *in X the closure of $\varphi(U)$ has an interior point, then $\varphi(U)$ is a neighbourhood of* 0 *in Y.*

Finally we remark that the class of G-spaces includes a class of topological linear spaces given in [1], namely spaces with réseau of type P.[2] In fact, we can prove that a topological linear space E is G-space if and only if there exists a monotone réseau in E which satisfies certain condition weaker than that of type P.

References

[1] M. De Wilde: Réseaux dans les espaces linéaires à semi-normes. Mémoires de la Société royale des sciences de Iiége. Ser. 5, Tome 18, Fasc. 2.

[2] M. Nakamura: On closed graph theorem. Proc. Japan Acad., **46** (8), 846–849 (1970).

2) M. De Wilde also defined, in [1], the class of spaces with réseau of type K and of type $\mathcal{E}$, but we can prove that, for réseau in general, being of type K or $\mathcal{E}$ is equivalent to being of type P.