# 23. *Approximate Propervalues and Characters of $C^*$-algebra*

By Isamu KASAHARA*) and Hiroshi TAKAI**)

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**1. Introduction.** Recently, Bunce [2] established a kind of reciprocity among the characters of singly generated $C^*$-algebras and approximate propervalues of the generators. He proved, among others, the following theorem:

**Theorem 1.** *If $A$ is a hyponormal operator acting on a Hilbert space $\mathfrak{H}$ and $\lambda$ is an approximate propervalue of $A$, then there is a character $\phi$ on the $C^*$-algebra $\mathfrak{A}$ generated by $A$ (and 1) such that*

(1) $$\phi(A)=\lambda.$$

In the above theorem, a *character* means a multiplicative state of $\mathfrak{A}$. A *state* $\phi$ is a positive linear functional on $\mathfrak{A}$ with $\phi(1)=1$, and $\phi$ is *multiplicative* if

(2) $$\phi(AB)=\phi(A)\phi(B),$$

for every $A, B \in \mathfrak{A}$.

Bunce [2] also proved the following theorem which is originally established by Arveson:

**Theorem 2.** *If $\lambda$ is a spectre of $A$ with $|\lambda|=\|A\|$, then there is a character $\phi$ on $\mathfrak{A}$ which satisfies* (1).

In the present note, we shall show that a kind of approximate propervalues has a closed connection with the characters of singly generated $C^*$-algebras. As consequences, the above mentioned theorems of Arveson and Bunce are proved under a unified method.

**2. Normal approximate propervalues.** In this note, we shall mean an operator $A$ is a bounded linear operator acting on $\mathfrak{H}$. Following after Halmos [4], we shall call a complex number $\lambda$ is an *approximate propervalue* of $A$ *provided that $\lambda$ and $A$* satisfy

(3) $$\|Ax_n-\lambda x_n\|\to 0 \qquad (n\to\infty)$$

for a sequence $\{x_n\}$ of unit vectors. Furthermore, if $\lambda$ and $A$ satisfy (3) and

(3*) $$\|A^*x_n-\lambda^*x_n\|\to 0 \qquad (n\to\infty),$$

then $\lambda$ is called a *normal approximate propervalue* of $A$ (in [7], $\lambda$ is called an approximate *reducing* propervalue). By the spectral theorem, we can see that every spectre of a normal operator is a normal approximate propervalue.

*) Momodani Senior Highschool, Osaka.
**) Department of Mathematics, Osaka Kyoiku University.

Let us denote by $\pi(A)$ the set of all approximate propervalues of $A$ and call it the *approximate spectrum* of $A$ which is a nonvoid compact set in the plane. Similarly, we can define the *normal approximate spectrum* as the set of all normal approximate propervalues of $A$. Unfortunately, there is an operator which has void normal approximate spectrum, as proved by Halmos [5]. However, Stampfli [7] established that the class $\bar{\boldsymbol{R}}_1$ of all operators having non-void normal approximate spectra coincides with the closure of all operators with a one-dimensional reducing subspace. He also proved that $\bar{\boldsymbol{R}}_1$ contains all hyponormal operators, compact operators and Toeplitz operators.

Our main result in this note is the following theorem:

**Theorem 3.** *If $\lambda$ is a normal approximate propervalue of $A$, then there is a character $\phi$ on the $C^*$-algebra $\mathfrak{A}$ generated by $A$ and 1 which satisfies* (1).

**Proof.** (3) implies at once there is a sequence $\{P_n\}$ of projections satisfying

(4) $$\|(A-\lambda)P_n\|\to 0 \qquad (n\to\infty)$$

and

(4*) $$\|(A-\lambda)^*P_n\|\to 0 \qquad (n\to\infty).$$

For example, put $P_n = x_n \otimes x_n$ where

(4) $$(x_n\otimes x_n)x=(x\,|\,x_n)x_n.$$

Let $\mathfrak{J}$ be the set of all operators in $\mathfrak{A}$ satisfying

$$\|BP_n\|\to 0, \qquad (n\to\infty).$$

We can easily conclude that $\mathfrak{J}$ is a proper left ideal of $\mathfrak{A}$. Hence by [3; § 2] there is a pure state $\phi$ of $\mathfrak{A}$ which satisfies

(6) $$\mathfrak{J}\subset \ker \phi,$$

where $\ker \phi = \{C \in \mathfrak{A} \,|\, \phi(C^*C)=0\}$.

Now we are in the position to tail the proof of Bunce [2; Proposition 9]. By (6), we have $\phi(A)=\lambda$ and $\phi(A^*)=\lambda^*$. Moreover we have

$$\phi(BA)=\phi(B)\lambda=\phi(B)\phi(A)$$

and

$$\phi(BA^*)=\phi(B)\lambda^*=\phi(B)\phi(A^*)$$

for every $B \in \mathfrak{A}$. If $p(A, A^*)$ is a polynomial in $A$ and $A^*$, we have

$$\phi(p(A,A^*))=p(\phi(A),\phi(A^*)).$$

Since the polynomials in $A$ and $A^*$ are dense in $\mathfrak{A}$, we can conclude that $\phi$ is a character of $\mathfrak{A}$.

**3. Applications.** We shall prove here Theorems 1 and 2.

Theorem 1 is clear by Theorem 3 and the following theorem which is due to Berberian [1]:

**Theorem 4.** *If $A$ is hyponormal, then every approximate propervalue is normal.*

**Proof.** $A$ is *hyponormal* if $AA^*\leqq A^*A$. Hence we can easily deduce that $A-\lambda$ is hyponormal too and $\|A^*x\|\leqq\|Ax\|$ for every $x\in\mathfrak{H}$. Hence if $\{x_n\}$ satisfies (3), then we have

$$\|(A-\lambda)^*x_n\|\leqq\|(A-\lambda)x_n\|\to 0 \qquad (n\to\infty),$$

so that (3) is satisfied. Therefore $\lambda$ is normal.

Theorem 2 is a consequence of Theorem 3 and the following known theorem, cf. [7]:

**Theorem 5.** *If $\lambda$ is a spectre of $A$ with $|\lambda|=\|A\|$, then $\lambda$ is a normal approximate propervalue.*

**Proof.** We need the following known fact: $\lambda\in\sigma(A)$ and $|\lambda|=\|A\|$ imply $\lambda\in\pi(A)$, cf. [4; Problem 63] and [6]. Suppose that $\{x_n\}$ is a sequence of unit vectors satisfying (3). Then we have

$$\|(A-\lambda)x_n\|^2=\|Ax_n\|^2-2\,\mathrm{Re}\,\lambda^*(Ax_n\,|\,x_n)+|\lambda|^2\to 0 \qquad (n\to\infty).$$

On the other hand, we have

$$|(Ax_n\,|\,x_n)-\lambda(x_n\,|\,x_n)|\leqq\|x_n\|\,\|Ax_n-\lambda x_n\|\to 0 \qquad (n\to\infty).$$

Hence we have

$$\begin{aligned}\|(A-\lambda)^*x_n\|^2&=\|A^*x_n\|^2-2\,\mathrm{Re}\,\lambda(A^*x_n\,|\,x_n)+|\lambda|^2\\ &\leqq|\lambda|^2-2\,\mathrm{Re}\,\lambda(Ax_n\,|\,x_n)^*+|\lambda|^2\to 2\,|\lambda|^2-2\,|\lambda|^2=0\\ &\qquad\qquad (n\to\infty).\end{aligned}$$

Therefore $\lambda$ is a normal approximate propervalue.

**Remark.** We can relax the hypothesis of Theorem 2. *If $\lambda\in W(A)$ and $|\lambda|=\|A\|$, then $\lambda$ is a normal approximate propervalue of $A$,* where $W(A)$ is the *numerical range* of $A$ defined by

$$W(A)=\{(Ax\,|\,x)\,|\,\|x\|=1\}.$$

This follows from a theorem of Wintner-Hilbert-Orland which states that $\lambda\in W(A)$ *and* $|\lambda|=\|A\|$ *imply* $\lambda\in\pi(A)$, cf. [6]. Hence, Theorem 2 becomes: *If $\lambda\in W(A)$ and $|\lambda|=\|A\|$, then there is a character $\phi$ on $\mathfrak{A}$ which satisfies* (1).

## References


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