43. Some Characterizations of σ- and Σ-Spaces

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M-space and σ-space are important generalizations of metric space into two different directions. (See [2], [9]. As for general terminologies and symbols in general topology, see [4]. All spaces in the following are at least T_1 except in the Definition, and all maps (=mappings) are continuous.) It is well-known that they not only represent two aspects of metrizability but also they combined together imply metrizability itself if the space is T_2. M^*-space is an interesting and useful generalization of M-space (due to [1]), and Σ-space (due to [3]) is interesting since it generalizes two different types of spaces, M^*- and (regular) σ-spaces at the same time and still has some nice properties. (A space Y is called a Σ-space if it has a sequence $\mathcal{V}_1, \mathcal{V}_2, \cdots$ of locally finite closed covers satisfying the following condition:

(Σ) If $y_n \in C(y, \mathcal{V}_n) = \cap \{V \mid y \in V \in \mathcal{V}_n\}$, $n=1, 2, \cdots$, then $\{y_n\}$ clusters).

We have characterized M^*-space and σ-space as follows.

Theorem 1. *Y is an M^*-space if and only if there is a perfect map from an M-space X onto Y.*

Theorem 2. *The following are equivalent for a regular space Y.*

i) *Y is a σ-space,*

ii) *there is a half-metric space (X, X') and a perfect map f from X onto Y such that $f(X') = Y$,*

iii) *there is a half-metric space (X, X') and a closed (continuous) map f from X onto Y such that $f(X') = Y$.*

Theorem 1 and the equivalence of i) and ii) in Theorem 2 were announced in [6], [7] and proved in [8]. As for the condition iii) in Theorem 2, it is obvious that ii) implies iii), and it is also easy to show by use of Theorem 1 of [10] that iii) implies i).

The main purpose of the present paper is to prove Theorem 3 in the following.

Definition. A pair (X, X') of a topological space X and its subspace X' is called a *half-M-space* if X has a sequence $\mathcal{U}_1, \mathcal{U}_2, \cdots$ of open covers such that

i) $\mathcal{U}_1 > \mathcal{U}_2^* > \mathcal{U}_2 > \mathcal{U}_3^* > \cdots$,

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ii) if $x \in X'$ and $x_n \in S(x, \mathcal{U}_n)$, $n=1, 2, \cdots$, then the point sequence $\{x_n\}$ has a cluster point in X.

Now the reader will agree with us upon that the following theorem is a quite natural conclusion to be compared with the previous two theorems, because half-M-space is a generalization of both M-space and half-metric space.

Remark. Precisely speaking, a half-metric space (X, X') is half-M provided X is normal. We may revise the definition of half-metric space in [6]–[8] as follows. A pair (X, X') of a topological space X and its subspace X' is called a half-metric space if X has a sequence $\mathcal{U}_1, \mathcal{U}_2, \cdots$ of open covers such that i) $\mathcal{U}_1 > \mathcal{U}_2^* > \cdots$, ii) for each $x \in X'$ and every nbd (=neighborhood) U of x in X, there is n for which $S(x, \mathcal{U}_n) \subset U$. Then every half-metric space in the revised sense is unconditionally half-M while Theorem 2 remains true for half-metric spaces in the revised sense.

Theorem 3. *Y is a Σ-space if and only if there is a half-M-space (X, X') and a perfect map f from X onto Y such that $f(X')=Y$.*

To prove this theorem we need the following lemma.

Lemma. *Y is a Σ-space if and only if there is a subspace X of a Baire's 0-dimensional metric space $N(A)$ and a multivalued map f from X onto Y such that*

i) *$f(x) \neq \emptyset$ for every $x \in X$,*

ii) *$f(F)$ is closed in Y for every closed set F of X,*

iii) *$f^{-1}(y)$ is a (non-empty) compact set for each $y \in Y$,*

iv) *for each $y \in Y$ there is $x \in f^{-1}(y)$ such that if $y_n \in f(S_{1/n}(x))$, $n=1, 2, \cdots$, then $\{y_n\}$ clusters in Y, where $S_\varepsilon(x)$ denotes the ε-nbd of x.*

Proof of Lemma. *Sufficiency.* Let $\{\mathcal{U}_n \mid n=1, 2, \cdots\}$ be a sequence of locally finite closed covers of X such that mesh $\mathcal{U}_n \to 0$. Then $\mathcal{V}_n = f(\mathcal{U}_n) = \{f(U) \mid U \in \mathcal{U}_n\}$, $n=1, 2, \cdots$ are locally finite closed covers of Y because of ii) and iii). Assume that $y_n \in C(y, \mathcal{V}_n)$, $n=1, 2, \cdots$ in Y. Then choose $x \in f^{-1}(y)$ satisfying iv) and also choose $U_n \in \mathcal{U}_n$, $n=1, 2, \cdots$ such that $x \in U_n$. Then $y_n \in f(U_n)$. Since diameter $U_n \to 0$, it follows from iv) that $\{y_n\}$ clusters. Thus Y is Σ.

Necessity. Let Y be a Σ-space with a sequence $\mathcal{V}_1, \mathcal{V}_2, \cdots$ of locally finite closed covers satisfying (Σ). Let $\mathcal{V}_n = \{V_\alpha \mid \alpha \in A_n\}$, $n=1, 2, \cdots$. We may index all $\mathcal{V}_n$ as $\mathcal{V}_n = \{V_\alpha^n \mid \alpha \in A\}$, where $A = \bigcup_{n=1}^\infty A_n$, and $V_\alpha^n = \emptyset$ for $\alpha \in A - A_n$. We may also assume that the intersections of any members of $\mathcal{V}_n$ belong to $\mathcal{V}_n$. Let $X = \{(\alpha_1, \alpha_2, \cdots) \in N(A) \mid V_{\alpha_1}^1 \cap V_{\alpha_2}^2 \cap \cdots \neq \emptyset\}$, where $N(A)$ denotes the Baire's 0-dimensional metric space with index set A, i.e. the countable product of the discrete space A. Define a multivalued map f from X to Y by $f(\alpha_1, \alpha_2, \cdots) = V_{\alpha_1}^1 \cap V_{\alpha_2}^2 \cap \cdots$ for $(\alpha_1, \alpha_2, \cdots) \in X$. Then i) is obviously satisfied. Since each

$\mathcal{V}_n$ is a locally finite closed cover, ii) and iii) can be proved in a similar way as in the proof of Theorem 1 of [5]. It is also easy to prove iv). Let $y \in Y$, then since $C(y, \mathcal{V}_n) \in \mathcal{V}_n$, we may let $C(y, \mathcal{V}_n) = V^n_{\alpha_n}$, $n=1, 2, \cdots$. Now $x = \alpha_1, \alpha_2, \cdots)$ is obviously a point in $f^{-1}(y)$ satisfying iv).

Proof of Theorem 3. *Sufficiency.* Let $\mathcal{U}_1, \mathcal{U}_2, \cdots$ be a sequence of open covers of X satisfying i), ii) in Definition. Then, as observed in [11], it follows from i) that for each i there is a locally finite open cover $\mathcal{V}_i$ of X with $\mathcal{V}_i < \mathcal{U}_i$. Let $\overline{\mathcal{V}}_i = \{\bar{V} \mid V \in \mathcal{V}_i\}$, $f(\overline{\mathcal{V}}_i) = \mathcal{W}_i$. Then $\{\mathcal{W}_i \mid i=1, 2, \cdots\}$ is easily seen to be a sequence of locally finite closed covers of Y satisfying (Σ). Hence Y is a Σ-space.

Necessity. Let f be a multivalued map from a metric space S onto Y satisfying i)–iv) of Lemma. Let Z be a compact T_2-space which contains S as a subspace. (There is such a space Z by virtue of Tychonoff's Theorem.) Then we define a subset X of the product space $Y \times S$ and its subset X' as follows.

$X = \{(y, s) \in Y \times S \mid y \in f(s)\}$,

$X' = \{(y, s) \in X \mid \text{if } y_n \in f(S_{1/n}(s)),\ n=1, 2, \cdots, \text{ then } \{y_n\}$ clusters in $Y\}$.

Furthermore we denote by π_S and π_Y the projections from X onto S and Y respectively. First we can prove that X is a closed set in $Y \times Z$. Let $(y, z) \in Y \times Z - X$; then since $f^{-1}(y)$ is a compact set of S by iii) of Lemma, it is closed in Z satisfying $z \notin f^{-1}(y)$. Hence there are open sets W and W' in Z such that $z \in W$, $f^{-1}(y) \subset W'$ and $W \cap W' = \emptyset$. By ii) of Lemma $V = Y - f(S - W')$ is an open nbd of y in Y. Therefore $V \times W$ is a nbd of (y, z) in $Y \times Z$. We claim that $V \times W$ is disjoint from X. To prove it, let $p = (v, w) \in V \times W$. If $w \notin S$, then $p \notin X$. If $w \in S$, then $w \in S - W'$, and hence $f(w) \cap V = \emptyset$. This implies that $v \notin f(w)$, and hence $p = (v, w) \notin X$. Therefore our claim is proved. Namely X is closed in $Y \times Z$.

Now, we can prove that (X, X') is a half-M-space. Let $\mathcal{V}_n$, $n=1, 2, \cdots$ be open covers of S with mesh $\mathcal{V}_n \to 0$ such that $\mathcal{V}_1 > \mathcal{V}_2^* > \cdots$. Then $\mathcal{U}_n = \pi_S^{-1}(\mathcal{V}_n)$, $n=1, 2, \cdots$ are open covers of X satisfying $\mathcal{U}_1 > \mathcal{U}_2^* > \cdots$. Let $x = (y, s) \in X'$, and $x_n = (y_n, s_n) \in S(x, \mathcal{U}_n)$, $n=1, 2, \cdots$ in X. Then $s_n \in S(s, \mathcal{V}_n)$, $n=1, 2, \cdots$ in S, and $y_n \in f(s_n)$. Hence by the definition of X', there is a cluster point y' of $\{y_n\}$. Now (y', s) is cluster point of $\{x_n\}$ in $Y \times S$. Since X is closed in $Y \times S$, $(y', s) \in X$. This proves that (X, X') is a half-M-space.

Finally we can prove that π_Y is a perfect map from X onto Y such that $\pi_Y(X') = Y$. $\pi_Y(X') = Y$ follows directly from iv) of Lemma and the definition of X'. π_Y is obviously continuous. For each $y \in Y$, $\pi_Y^{-1}(y)$ is homeomorphic to $f^{-1}(y)$ which is compact by iii) of Lemma. Thus the only thing we have to prove is that for every closed set C of X, $\pi_Y(C)$ is closed in Y. Since X is closed in $Y\times Z$, so is C. Let $y\in Y-\pi_Y(C)$. Then for each $z\in Z$, there are open nbds U_z of y and $V(z)$ of z such that $(U_z\times V(z))\cap C=\emptyset$. Cover the compact set $\{y\}\times Z$ with $U_{z_1}\times V(z_1)$, $\cdots$, $U_{z_k}\times V(z_k)$. Then $U=U_{z_1}\cap\cdots\cap U_{z_k}$ is a nbd of y disjoint from $\pi_Y(C)$. Thus $\pi_Y(C)$ is closed in Y proving Theorem 3.

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