67. *Spinnable Structures on Differentiable Manifolds*

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In this note we shall introduce a new structure, called a spinnable structure, on a differentiable manifold. Roughly speaking, a differentiable manifold is spinnable if it can spin around an axis as if the top spins. Such structure originated in connection with the study of foliations (Lawson [1], Tamura [4], [5]), and the existence of a codimension-one foliation follows from a spinnable structure of a special kind.

Definition 1. An m-dimensional differentiable manifold M^m is called *spinnable* if there exists an $(m-2)$-dimensional submanifold X satisfying the following conditions:

(i) The normal bundle of X is trivial.

(ii) Let $X\times D^2$ be a tubular neighborhood of X, then $C=M^m-X\times \operatorname{Int} D^2$ is the total space of a fibre bundle ξ over a circle.

(iii) Let $p: C\to S^1$ be the projection of ξ, then the diagram

$$\begin{tikzcd} X\times S^1 \arrow[r, "\iota"] \arrow[dr, "p'"'] & C \arrow[d, "p"] \\ & S^1 \end{tikzcd}$$

commutes, where ι denotes the inclusion map and p' denotes the natural projection onto the second factor.

The submanifold X is called an *axis* and a fibre F of ξ is called a *generator*. F is an $(m-1)$-dimensional submanifold of M^m. Obviously $\partial F=X$ holds if $\partial M^m=\emptyset$. The fibre bundle $\xi=\{C, p, S^1, F\}$ is called a *spinning bundle* and the pair (X,ξ) is called a *spinnable structure*[1] of M^m.

Example 1. S^m is spinnable with (naturally imbedded) S^{m-2} as axis. $\boldsymbol{R}^m$ is spinnable with $\boldsymbol{R}^{m-2}$ as axis.

Remark 1. It is easy to see that, if M^m is spinnable, then the decomposition $M=(F\times I)\cup(F\times I)$ holds.

The following theorem is easily proved.

Theorem 1. *Suppose that M_1^m and M_2^m are spinnable connected differentiable manifolds having connected submanifolds X_1 and X_2 as*

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1) The same concept is called a fibered knot in a recent paper of A. Durfee and H. Lawson, Fibered knots and foliations of highly connected manifolds.

axes respectively. Then the connected sum $M_1^m \# M_2^m$ admits spinnable structure having $X_1 \# X_2$ as axis.

Example 2. The Milnor fibering gives various spinnable structures on S^{2n+1} (Milnor [2]).

Example 3. S^{4n+1} ($n \geqq 2$) has $S^{2n} \times S^{2n-1}$ as an axis (Tamura [4]).

Problem 1. Determine the axes of S^m.

Theorem 2. *Let M^{2n+1} be a simply connected closed $(2n+1)$-dimensional differentiable manifold ($n \geqq 3$) such that $H_n(M; Z)$ is torsion free, then M^{2n+1} is spinnable.*

Sketch of proof. Let f be a nice function on M^{2n+1} with minimal critical points (Smale [3]) and let $W = f^{-1}([0, n+1/2])$, $W' = f^{-1}([n+1/2, 2n+1])$. Then $M^{2n+1} = W \cup W'$. It follows by the homology exact sequence of $(W, \partial W)$ that $\iota_*: H_q(\partial W; Z) \to H_q(W; Z)$ is isomorphic for $0 \leqq q \leqq n-1$, where $\iota: \partial W \to W$ is the inclusion map. Further $H_n(\partial W; Z) \cong H^n(W; Z) \oplus H_n(W; Z)$. Since ∂W is a simply connected closed $2n$-dimensional differentiable manifold ($n \geqq 3$), there exists a nice function g on ∂W with minimal critical points. Let F be a compact $2n$-dimensional submanifold of ∂W obtained from $g^{-1}([0, n-1/2])$ by adding n-handles corresponding to generators of $H_n(W; Z) \subset H_n(\partial W; Z)$ and of the torsion subgroup of $H_{n-1}(\partial W; Z)$. Then the inclusion map $F \to W$ is a homotopy equivalence. Further, since $H_n(M; Z)$ is torsion free, it can be shown that the inclusion map $F \to W'$ is a homotopy equivalence. Thus $W = F \times I$, $W' = F \times I$ by the h-cobordism theorem (Smale [3]). This observation implies that M^{2n+1} admits a spinnable structure having ∂F as axis and F as generator.

Similar arguments yield the following theorem.

Theorem 3. *Let M^{2n} be a simply connected closed $2n$-dimensional differentiable manifold ($n \geqq 4$) such that $H_n(M; Z)$ is torsion free. Further, in case where n is even, assume that the matrix of intersection numbers of $H_n(M; Z)$ is even and its signature is zero. Then M^{2n} is spinnable.*

Remark 2. Hypothesis on $H_n(M; Z)$ in Theorem 2 may be replaced by some condition about the pairing $T \otimes T \to Q/Z$ of the torsion subgroup T of $H_n(M; Z)$.

Remark 3. It is easy to see that the index of a spinnable closed differentiable manifold is zero.

Remark 4. The complex projective space $P^m(C)$ of complex dimension m is spinnable if and only if m is odd.

Problem 2. Find a necessary and sufficient condition in order that a differentiable manifold be spinnable.

Definition 2. An m-dimensional differentiable manifold is called *specially spinnable* if it has S^{m-2} as an axis.

Remark 5. Every specially spinnable differentiable manifold is obtained from a fibering over S^1 by performing a surgery on a cross section.

Theorem 4. *Let M^{2n+1} be an $(n-1)$-connected closed $(2n+1)$-dimensional differentiable manifold $(n\geqq 3)$ such that $H_n(M;Z)$ is torsion free, then M^{2n+1} is specially spinnable.*

Sketch of proof. It follows by the arguments as in the proof of Theorem 1 that there exists a generator F of M^{2n+1} such that $F=E_1 \natural E_2 \natural \cdots \natural E_r$, where r denotes the n-th Betti number of M^{2n+1} and E_i denotes the total space of a D^n-bundle over S^n $(i=1,2,\cdots,r)$. In case where $n\equiv 3,5,6,7 \pmod 8$, we may assume that each (E_i, p_i, S^n, D^n) is trivial. Let N denote a tubular neighborhood of the diagonal of $S^n\times S^n$. Then N is the total space of a D^n-bundle over S^n. Notice that the boundary of a differentiable manifold $E_i \veebar N$ obtained by plumbing $E_i=S^n\times D^n$ with N is S^{2n-1}. Now we can construct from F a new generator $\hat{F}=(E_1\veebar N)\natural(E_2\veebar N)\natural\cdots\natural(E_r\veebar N)$. (Compare the arguments in Tamura [4]). Since $\partial\hat{F}=S^{2n-1}$, M^{2n+1} is specially spinnable. In case where $n\equiv 0,2,4 \pmod 8$, we can construct a new generator $\hat{F}$ such that $\partial\hat{F}=S^{2n-1}$ from F by 'plumbing' E_i with W, where W denotes the compact $2n$-dimensional manifold constructed in a previous paper (Tamura [4], section 2). Similar arguments are applicable for the case $n\equiv 1 \pmod 8$.

Problem 3. Find a necessary and sufficient condition in order that a differentiable manifold be specially spinnable.

As is well known, the total space of a fibre bundle over S^1 has a codimension-one foliation compatible with the fibering (Lawson [1], Lemma 1). Thus the following theorem holds.

Theorem 5. *Let M be a spinnable differentiable manifold with axis X. Suppose that $X\times D^2$ has a codimension-one foliation such that $X\times S^1$ is a sum of leaves. Then M has a codimension-one foliation.*

The following theorem is a direct consequence of the existence of codimension-one foliations of odd dimensional spheres (Tamura [4], [5] Lemma).

Theorem 6. *Every specially spinnable odd dimensional differentiable manifold has a codimension-one foliation.*

By combining Theorems 4 and 6, we obtain the following theorem without using any classification theorem of differentiable manifolds.

Theorem 7. *Every $(n-1)$-connected torsion free closed $(2n+1)$-dimensional differentiable manifold $(n\geqq 3)$ has a codimension-one foliation.*

Detailed proofs of the above theorems will appear elsewhere.
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References

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