109. *Structure of Left QF-3 Rings*

By Toyonori KATO
College of General Education, Tohoku University, Kawauchi, Sendai

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The purpose of this note is to establish a structure theorem for left QF-3 rings, an analogue to one for QF-3 algebras by Morita [14], introducing a new notion of left QF-3 rings.

It turns out that not only faithful projective-injective modules but also dominant modules play a vital role in the structure theory of left (-right) QF-3 rings.

Throughout this note, rings R and S will have identity and modules will be unital. ${}_SX$ will signify the fact that X is a left S-module. We adopt the notational convention of writing module-homomorphism on the side opposite the scalars.

Definition (Kato [10]). A module P_R is called dominant if P_R is faithful finitely generated projective and ${}_SP$ is lower distinguished[1] with $S=\mathrm{End}(P_R)$.

The following definition of left QF-3 rings finds no mention in the literature.

Definition. A ring R will be called left QF-3 if R contains idempotents e and f such that Re is a faithful injective left ideal and fR is a dominant right ideal.

Lemma 1[2]. *If e and f are idempotents of R such that ${}_RRe$ is injective and fR_R is faithful, then*

(1) $Re=\mathrm{Hom}({}_{fRf}fR, {}_{fRf}fRe)$, so $eRe=\mathrm{End}({}_{fRf}fRe)$.

(2) *${}_{fRf}fRe$ is injective.*

Proof. This is Proposition 2.1 of Tachikawa [25].

Lemma 2. *The double centralizer of any faithful torsionless right R-module is a left quotient*[3] *ring of R.*

Proof. See Colby and Rutter [3, 4], Tachikawa [25], Faith [5], and Kato [11].

Lemma 3. *Let ${}_SV$ be a cogenerator and $T=\mathrm{End}({}_SV)$. Then ${}_SV$ is linearly compact if and only if V_T is injective; then a module ${}_SU$ is linearly compact if and only if ${}_SU$ is V-reflexive.*

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1) ${}_SP$ is lower distinguished if ${}_SP$ contains a copy of each simple module. Cf. Azumaya [1].

2) Cf. Kato [13].

3) Q is a (the maximal) left quotient ring of R if Q is a ring extension of R and ${}_RQ$ is a (the maximal) rational extension of ${}_RR$. Cf. Findlay and Lambek [6].

Proof. This is Corollaries 1 and 2 of Onodera [19]. Cf. Müller [17] or Sandomierski [23].

Lemma 4. *Let P_R be a dominant module. Then $E(R_R)$, the injective hull of R_R, is torsionless if and only if P_R is injective.*

Proof. This is Lemma 1 of Kato [12]. Cf. Onodera [18].

Structure theorem. *Let S be a ring, ${}_SV$ an injective cogenerator,*

$$ {}_SU = {}_SV \oplus {}_SX = {}_SS \oplus {}_SY $$

with the projections $e\colon {}_S\mu \to {}_SV,\ {}_SU \to {}_SS$, and $Q = \mathrm{End}({}_SU)$. *Let R be a subring of Q containing* 1, *Qe and fQ. Then ${}_RRe$ is faithful injective and fR_R is dominant; R is thus a left* QF-3 *ring. Conversely, any left* QF-3 *ring R (containing idempotents e and f such that ${}_RRe$ is faithful injective and fR_R is dominant) is just obtained in this manner. Moreover,*

(1) Q is not only the maximal left, but also a right quotient ring of R, and $R = Q$ if and only if dom[4] ${}_RR \geqq 2$.

(2) ${}_RRe$ is dominant if and only if V_T is lower distinguished with $T = \mathrm{End}({}_SV)$.

(3) fR_R is injective if and only if ${}_SU$ is linearly compact.

Proof. The module U forms a ring (not necessarily with identity) under a multiplication

$$ (s+y)u = su \qquad \text{for } s \in S, y \in Y, u \in {}_SU. $$

Clearly U is a right faithful ring and an S-U-bimodule, so U is a subring of Q. It now follows from the identification $U \subset Q$ that

$$ U = fQ, \quad S = fQf, \quad V = fQe. $$

Thus ${}_{fRf}fRe = {}_{fQf}fQe = {}_SV$ is an injective cogenerator and $R \subset Q = \mathrm{End}({}_SU) = \mathrm{End}({}_{fQf}fQ) = \mathrm{End}({}_{fRf}fR)$, so ${}_{fRf}fRe \subset {}_{fRf}fR$ is a cogenerator (so necessarily lower distinguished) and fR_R is faithful. Hence fR_R is dominant. On the other hand, since $Q = \mathrm{End}({}_{fRf}fR)$ and ${}_{fRf}fRe$ is injective,

$$ {}_RRe = {}_RQe = {}_R\mathrm{Hom}({}_{fRf}fR, {}_{fRf}fRe) $$

is injective by Cartan and Eilenberg [2, Proposition 1.4, p. 107]. Moreover, ${}_{fRf}fR \subsetneq \prod {}_{fRf}fRe$ (recall that ${}_{fRf}fRe$ is a cogenerator) whence

$$ {}_RR \subset {}_RQ = {}_R\mathrm{Hom}({}_{fRf}fR, {}_{fRf}fR) \subsetneq \prod {}_R\mathrm{Hom}({}_{fRf}fR, {}_{fRf}fRe) = \prod {}_RRe\ , $$

so ${}_RRe$ is faithful. We thus conclude that R is a left QF-3 ring.

Conversely, let R be a left QF-3 ring with idempotents e and f such that ${}_RRe$ is faithful injective and fR_R is dominant. Let

$$ S = fRf, \quad {}_SV = {}_{fRf}fRe, \quad {}_SU = {}_{fRf}fR, $$
$$ {}_SX = {}_{fRf}fR(1-e), \quad {}_SY = {}_{fRf}fR(1-f), \quad Q = \mathrm{End}({}_SU), $$

then

$$ {}_SU = {}_SV \oplus {}_SX = {}_SS \oplus {}_SY $$

with the projections $e\colon {}_SU \to {}_SV$ and $f\colon {}_SU \to {}_SS$ (since fR_R is faithful). By Lemma 1 ${}_SV = {}_{fRf}fRe$ is injective. Moreover, ${}_RR \subsetneq \prod {}_RRe$ (since

4) Cf. Tachikawa [25] or Kato [9].

${}_R Re$ is faithful) whence

$${}_{fRf}fR = {}_{fRf}\mathrm{Hom}({}_R Rf, {}_R R) \hookrightarrow \prod {}_{fRf}\mathrm{Hom}({}_R Rf, {}_R Re) = \prod {}_{fRf}fRe,$$

so $\prod {}_S V = \prod {}_{fRf}fRe$ is an injective cogenerator (recall that fR_R is dominant) by Osofsky [20, Lemma 1], and hence, so is ${}_S V$ by Sugano [24, Lemma 1]. Now, R is a subring of Q since fR_R is faithful, $fR = fQ$ since $Q = \mathrm{End}({}_{fRf}fR)$, and $Re = Qe$ by Lemma 1.

(1) Since Q is the double centralizer of the dominant right ideal fR, Q is a left quotient ring of R by Lemma 2 and

$$\text{dom. dim } {}_Q Q \geqq 2$$

according to Kato [9, Theorem 2] (recall that ${}_{fRf}fR$ is a generator-cogenerator). Hence Q is the maximal left quotient ring of R by Tachikawa [25, Proposition 1.3]. On the other hand, since ${}_Q Qe$ is faithful and $Qe = Re$,

$$Q \subset \mathrm{End}(Qe_{eQe}) = \mathrm{End}(Re_{eRe})$$

is also a right quotient ring of R in view of Lemma 2. Now, $R = Q$ if and only if dom. dim ${}_R R \geqq 2$ again by Tachikawa [25, Proposition 1.3].

(2) By Lemma 1

$$T = \mathrm{End}({}_S V) = \mathrm{End}({}_{fRf}fRe) = eRe.$$

If ${}_R Re$ is dominant, then

$$Re_{eRe} \hookrightarrow \prod fRe_{eRe} = \prod V_T$$

(since fR_R is faithful) is lower distinguished, and hence, so is V_T. Conversely, if V_T is lower distinguished, then

$$V_T = fRe_{eRe} \subset Re_{eRe}$$

is lower distinguished, so ${}_R Re$ is dominant (since ${}_R Re$ is faithful).

(3) Let

$$T = \mathrm{End}({}_S V) = \mathrm{End}({}_{fRf}fRe) = eRe.$$

If fR_R is injective, it then follows from Lemma 1 that

$$fR = \mathrm{Hom}(Re_{eRe}, fRe_{eRe})$$

and $V_T = fRe_{eRe}$ is injective. According to Lemma 3, ${}_S U$ is thus linearly compact, since ${}_S U = {}_{fRf}fR$ is V-reflexive. Conversely, if ${}_S U$ is linearly compact, then so is ${}_S V$ (since ${}_S V$ is a submodule of ${}_S U$). Hence $fRe_{eRe} = V_T$ is injective and ${}_{fRf}fR = {}_S U$ is fRe-reflexive by Lemma 3. Thus

$$fR_R = \mathrm{Hom}(\mathrm{Hom}({}_{fRf}fR, {}_{fRf}fRe)_{eRe}, fRe_{eRe})_R$$
$$= \mathrm{Hom}(Re_{eRe}, fRe_{eRe})_R$$

is injective.

Corollary 1.[5] *Let S be a ring with a Morita duality*[6] ${}_S V$,

$${}_S U = {}_S V \oplus {}_S X = {}_S S \oplus {}_S Y$$

a V-reflexive module with the projections $e: {}_S U \to {}_S V$ and $f: {}_S U \to {}_S S$, and $Q = \mathrm{End}({}_S U)$. Let R be a subring of Q containing 1, *Qe and fQ.*

5) Cf. Morita [14] or Morita and Tachikawa [15].

6) ${}_S V$ is a Morita duality if ${}_S V$ and V_T are injective cogenerators with $T = \mathrm{End}({}_S V)$ and $S = \mathrm{End}(V_T)$. Cf. Sandomierski [22].

Then $_RRe$ and fR_R are injective dominant; R is thus a left-right QF-3 *ring. Conversely, any left-right* QF-3 *ring R (containing idempotents e and f such that $_RRe$ and fR_R are dominant) is just obtained in this manner. Moreover, Q is the maximal left-right quotient ring of R.*

Proof. From the preceding arguments, $_RRe$ and fR_R are injective dominant. Conversely, let R be a left-right QF-3 ring with idempotents e and f such that $_RRe$ and fR_R are dominant. According to Lemma 4, the dominant modules $_RRe$ and fR_R are injective since R is left-right QF-3. From the preceding arguments again, it now follows that $_{fRf}fRe$ and fRe_{eRe} are injective cogenerators and

$$Re=\mathrm{Hom}\,({}_{fRf}fR, {}_{fRf}fRe), \quad \text{so} \quad eRe=\mathrm{End}\,({}_{fRf}fRe),$$

$$fR=\mathrm{Hom}\,(Re_{eRe}, fRe_{eRe}), \quad \text{so} \quad fRf=\mathrm{End}\,(fRe_{eRe}).$$

Thus $_SV={}_{fRf}fRe$ is a Morita duality and $_SU={}_{fRf}fR$ is V-reflexive. Finally, Q is the maximal left-right quotient ring of R (cf. Colby and Rutter [4] and Müller [16]).

Definition. A subring S of R will be called left dominant if $S=fRf$ with fR $(f=f^2\in R)$ a dominant right ideal.

Corollary 2. (1) *Any ring* (*with* 1) *is a left dominant subring of a left* QF-3 *ring.*

(2)[7] *S is a ring with a left Morita duality if and only if S is a left dominant subring of a left-right* FQ-3 *ring.*

Example. Any minimal faithful[8] module P_R is dominant (see Colby and Rutter [3, Theorem 1], Fuller [7, Theorem 2.1] and Kato [13, Theorem 4]).

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7) Cf. Roux [21].

8) P is minimal faithful if P is faithful and is a direct summand of each faithful module. Cf. Thrall [26] or Jans [8].

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