107. A Note on Cogenerators in the Category of Modules

By Yutaka KAWADA

(Comm. by Kenjiro SHODA, M. J. A., Sept. 12, 1972)

Let A be a ring with identity and ${}_AW$ a cogenerator in the category of unitary left A-modules, and denote by $B=\mathrm{End}\,({}_AW)$ the endomorphism ring of ${}_AW$. Then W is regarded as an A-B-bimodule. As for the structure of ${}_AW$ in general, there was a useful result of Osofsky [5, Lemma 1]. As for the structure of W_B, recently Onodera has obtained an interesting result [4, Theorem 1].

The purpose of this paper is to establish the following two theorems:

Theorem 1. *Let ${}_AW$ be a cogenerator, and let $B=\mathrm{End}\,({}_AW)$ and $C=\mathrm{End}\,(W_B)$. Then W_B is absolutely pure and semi-injective. Furthermore A is dense in C relative to the finite topology. In particular, if ${}_AW$ is finitely cogenerating in the sense of Morita* [3], *then ${}_AW$ possesses the double centralizer property, i.e. $C=A$.*

Theorem 2. *Let ${}_AW$ be a cogenerator and $B=\mathrm{End}\,({}_AW)$, and denote by $S(W_B)$ the socle of W_B. Let further $\{V_\lambda \mid \lambda\in\Lambda\}$ be a complete representative system of isomorphism classes of simple left A-modules such that $E(V_\lambda)\subset W$ for each $\lambda\in\Lambda$ (Cf.* [5, *Lemma* 1]), *where $E(V_\lambda)$ denotes an injective hull of V_λ. Then $S(W_B)\subset' W_B$, and*

$$S(W_B)=\sum_{\lambda\in\Lambda}\oplus V_\lambda B$$

is the decomposition of $S(W_B)$ into homogeneous components.

Throughout this paper, all modules are assumed to be unitary, and we shall keep above notations and meanings. In particular, ${}_AW$ denotes always a cogenerator and B (resp. C) denotes the endomorphism ring of ${}_AW$ (resp. of W_B).

1. Proof of Theorem 1.

Previous to this, we need some lemmas.

Lemma 1 [4, Theorem 1]. *Let M be a left A-module and set $M_B^*=\mathrm{Hom}_A({}_AM, {}_AW_B)$. Then, for each finitely generated B-submodule U of M_B^* and for each B-homomorphism $f: U_B\to W_B$, there exists an element v in M such that $f=\rho(v)\cdot i$, where $i: U_B\to M_B^*$ implies the inclusion map and $\rho: M\to \mathrm{Hom}_B(M_B^*, W_B)$ is the canonical map defined by $\rho(x)(g)=g(x)$ for every $x\in M$ and $g\in M^*$.*

Let us denote by W^n (resp. B^n) the direct sum of n copies of W (resp. of B). For a subset X of W^n, set

$$(0:X)_{B^n}=\{(b_1,\cdots,b_n)\in B^n \mid \textstyle\sum v_ib_i=0 \quad \text{for all } (v_1,\cdots,v_n)\in X\}.$$

Similarly for a subset Y of B^n, set

$(0:Y)_{W^n}=\{(v_1,\cdots,v_n)\in W^n \mid \sum v_ib_i=0$ for all $(b_1,\cdots,b_n)\in Y\}$.
Then the following is a direct consequence of [4, Proposition 4].

Lemma 2. *For any A-submodule U of ${}_AW^n$ the annihilator relation holds:*

$$(0:(0:U)_{B^n})_{W^n}=U.$$

Lemma 3. *Any A-submodule U of ${}_AW^n$ becomes a C-submodule, that is, $U=C\cdot U$.*

Proof. Since ${}_AW$ is faithful we may regard as $A\subset C$. $C\cdot U$ is thereby an A-submodule of W^n and obviously $(0:U)_{B^n}=(0:C\cdot U)_{B^n}$. Hence we get $U=C\cdot U$ by Lemma 2.

Recall now that a right B-module N is said to be absolutely pure (resp. semi-injective), if for each finitely generated B-submodule Y of B^n_B, n arbitrary, (resp. of N_B) and for each B-homomorphism $f:Y\to N$ there exists a B-homomorphism $g:B^n\to N$ (resp. $g:N\to N$) such that $g\mid Y=f$. We are in a position to prove Theorem 1:

Proof of Theorem 1. At first set $M={}_AW^n$ (resp. $M={}_AA$) in Lemma 1. Then it is seen that W_B is absolutely pure (resp. semi-injective). Next we want to show that A is dense in C relative to the finite topology. Let $v_1,\cdots,v_n$ be given elements of W and let c_0 be an element of C. Since by Lemma 3 $C(v_1,\cdots,v_n)=A(v_1,\cdots,v_n)$ in W^n, there exists an element a_0 of A such that $c_0(v_1,\cdots,v_n)=a_0(v_1,\cdots,v_n)$, i.e. $c_0v_i=a_0v_i$ for $i=1,\cdots,n$. This implies that A is dense in C relative to the finite topology.

Finally assume that ${}_AW$ is finitely cogenerating. Since ${}_AW$ is faithful and finitely cogenerating, there are elements $v_1,\cdots,v_n$ in W such that $\cap(0:v_i)_A=0$. Let c_0 be a given element of C and v_0 an element of W. Then, since by Lemma 3 $C(v_1,\cdots,v_n,v_0)=A(v_1,\cdots,v_n,v_0)$ in W^{n+1}, there is an element a_0 of A such that

$$c_0(v_1,\cdots,v_n,v_0)=a_0(v_1,\cdots,v_n,v_0) \quad \text{and so} \quad c_0v_0=a_0v_0.$$

Similarly, for another element v of W we get an element a of A such that

$$c_0(v_1,\cdots,v_n,v)=a(v_1,\cdots v_n,v) \quad \text{and so} \quad c_0v=av.$$

But, since $a_0(v_1,\cdots,v_n)=c_0(v_1,\cdots,v_n)=a(v_1,\cdots,v_n)$ in W^n we have $a_0v_i=av_i$ for $i=1,\cdots,n$ and hence $a_0-a\in\cap(0:v_i)_A=0$. Thus it is shown that $c_0v=a_0v$ for every $v\in W$, that is, $C\subset A$ and consequently $C=A$. Hence the proof of Theorem 1 is completed.

2. Proof of Theorem 2.

Previous to this, we shall recall the following

Lemma 4 [5, Lemma 1]. *${}_AW$ is a cogenerator if and only if W contains a copy of the injective hull $E({}_AV)$ of each simple left A-module V.*

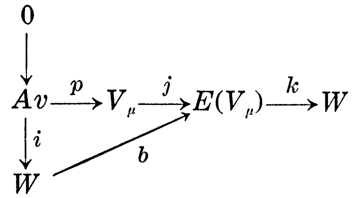
By Lemma 4 there exists a complete representative system

$\{V_\lambda \mid \lambda \in \Lambda\}$ of isomorphism classes of simple left A-modules such that $E(V_\lambda) \subset W$ for each $\lambda \in \Lambda$.

Proof of Theorem 2. We shall proceed step by step.

Step 1. $\sum V_\lambda B \subset' W_B$.

Let v be any non-zero element of W and let $V_\mu (\mu \in \Lambda)$ be a simple quotient of Av. Then, since $E(V_\mu)$ is injective, there exists an element b of B such that the following diagram is commutative:



where i, j and k are inclusion maps and p is an epimorphism. Therefore $p(v) = vb$, i.e $V_\mu = Avb$ and hence $vB \cap \sum V_\lambda B \neq 0$. This shows that $\sum V_\lambda B \subset' W_B$.

Step 2. $S(W_B) = \sum V_\lambda B$.

Let vB be any simple B-submodule of W and let $V_\mu (\mu \in \Lambda)$ be a simple quotient of Av. Then, in the same way as in Step 1, there is an element b of B such that $V_\mu = Avb$ and so $vB = vbB \subset V_\mu B$. This implies that $S(W_B) \subset \sum V_\lambda B$.

Conversely, assume that $vb_0 \neq 0$ $(v \in V_\lambda, b_0 \in B)$. Since $Avb_0 \cong Av = V_\lambda$ by the map $q: xvb_0 \to xv$ $(x \in A)$, in a similar way as in Step 1, we get an element b of B such that $v = q(vb_0) = vb_0 b$ and hence $vB = vb_0 B$. This implies that vB is simple for every $v \neq 0$ in V_λ and hence $V_\lambda B \subset S(W_B)$. Therefore $\sum V_\lambda B \subset S(W_B)$ and consequently $S(W_B) = \sum V_\lambda B$.

Step 3. $V_\lambda B$ $(\lambda \in \Lambda)$ is a homogeneous component of $S(W_B)$.

Take now a non-zero element, say v_λ, in V_λ. Then, as has been proved above, $v_\lambda B$ is a simple right B-module. Assume that U is a simple B-submodule of W such that $v_\lambda B \cong U$. Since W_B is semi-injective and since A is dense in C relative to the finite topology by Theorem 1, there exists an element a of A such that $U = av_\lambda B$.

Conversely, for any element a_0 of A we have either $a_0 v_\lambda B \cong v_\lambda B$ or $a_0 v_\lambda B = 0$, because $v_\lambda B$ is simple. Therefore the homogeneous component of $S(W_B)$ including $v_\lambda B$ coincides with $V_\lambda B$.

Step 4. $S(W_B) = \sum \oplus V_\lambda B$.

Notice that if $\lambda \neq \mu$ then $v_\lambda B \ncong v_\mu B$, where v_λ (resp, v_μ) is a non-zero element of V_λ (resp. V_μ). Because, if $v_\lambda B \cong v_\mu B$ then, in the same way as in Step 3, there is an element a of A such that $v_\mu B = av_\lambda B$. Hence $v_\mu = av_\lambda b$ for some b in B, whence it follows that $V_\mu = V_\lambda b \cong V_\lambda$, a contradiction.

Therefore, if $\lambda \neq \mu$ then the type of $V_\lambda B$ is different from one of

$V_\mu B$. Consequently $S(W_B)=\sum \oplus V_\lambda B$ (Cf. [1, p. 80]). Thus the proof of Theorem 2 is completed.

The following is a direct consequence of Theorems 1 and 2.

Corollary 1. *Let A be a left cogenerator ring. Then A_A is absolutely pure and $S(A_A)\subset' A_A$.*

Corollary 2. *Assume that ${}_AW$ is an injective cogenerator. Then $S({}_AW)=S(W_B)$ where $B=\mathrm{End}\,({}_AW)$.*

Proof. Obviously by Theorem 2 $S(W_B)=\sum V_\lambda B\subset S({}_AW)$. Conversely assume that V is a simple A-submodule of W. Then $V\cong V_\mu$ for some $\mu\in\Lambda$, and by injectivity of ${}_AW$ we can find an element b of B such that $V=V_\mu b$. Hence $S({}_AW)\subset\sum V_\lambda B=S(W_B)$ and consequently $S({}_AW)=S(W_B)$.

References

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