103. The Asymptotic Formulas for Eigenvalues of Elliptic Operators which Degenerate at the Boundary

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1. Introduction and Theorem.

The purpose of this paper is to derive asymptotic formulas with remainder estimates for the distribution of eigenvalues of elliptic operators which degenerate on the boundary. The formula without remainder estimate was established by I. A. Solomešč [6]. We investigate our problem under assumptions similar to those of [6]. Only the theorem and an outline of its proof are presented here and the details will be published elsewhere.

Let Ω be a bounded domain of $\boldsymbol{R}^n$ having the restricted cone property ([1]). Let $\{O_i\}$ and $\{C_i\}$ be the covering of $\partial\Omega$ and the set of corresponding cones, respectively, as guaranteed by the restricted cone property. Let ξ be any unit vector which is a positive multiple of the vector in the cone C_i. Then we assume that there is a constant K_0 such that

$$\delta(x_0+t\xi)\geq K_0\{\delta(x_0)+t\}$$

for any $x_0\in O_i$ and $0\leq t\leq h_i$, where h_i is the height of C_i and $\delta(x)=\mathrm{dist}\,(x,\partial\Omega)$.

The set of complex-valued functions $f\in C^{m*}(\Omega)$ having a finite integral

$$\|u\|_{m,a}^2=\int_\Omega \delta(x)^a \sum_{|\alpha|\leq m}|D^\alpha f|^2dx$$

is denoted by $C^{m*}(\Omega)$ and the closure of $C^{m*}(\Omega)$ and $C_0^\infty(\Omega)$ with respect to the norm $\|\ \|_{m,a}$ is denoted by $W_{m,a}(\Omega)$ and $\mathring{W}_{m,a}(\Omega)$ respectively. Let V be some closed subspace of $W_{m,a}(\Omega)$ containing $\mathring{W}_{m,a}(\Omega)$ and B be an integro-differential sesquilinear form of order m

$$B[u,v]=\int_\Omega \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x)D^\alpha u\overline{D^\beta v}\,dx+B_1[u,v]$$

satisfying

$$\mathrm{Re}\,B[u,u]\geq\delta\|u\|_m^2 \qquad \text{for any } u\in V,$$

and

$$|B_1[u,v]|\leq K_1\{\|u\|_{m,a}\|v\|_{m-1,a}+\|u\|_{m-1,a}\|v\|_{m,a}\} \qquad \text{for } u,v\in V$$

where δ and K_1 are some positive constants. For the coefficients we shall assume that they are symmetric (i.e. $a_{\alpha\beta}(x)=\overline{a_{\beta\alpha}(x)}$), belong to $C^\infty(\Omega)$ and there is a positive constant K_2 such that

$$K_2^{-1} \leq \Big| \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x)/\delta(x)^a \Big| \leq K_2$$

for any $x \in \Omega$. Moreover it is assumed that $2m > n$ and $0 < an/2m < 1$.

Now, as in the previous paper [2], [3], we consider an operator A such that

$$B[u, v] = (Au, v) \qquad \text{for any } u, v \in V$$

where the bracket on the right denotes the pairing between V^* (=the antidual space of V) and V. We denote $\{\lambda_j\}_{j=0}^{\infty}$ the eigenvalues of the operator A and put $N(t) = \sum_{\mathrm{Re}\,\lambda_j < t} 1$.

Theorem. *Under the hypotheses stated above we have*

$$N(t) = C_0 t^{n/2m} + O(t^{(n-\tau)/2m})$$

as $t \to \infty$ *where*

$$C_0 = \frac{\sin(n\pi/2m)}{n\pi/2m} \int_\Omega C_0(x)dx$$

$$C_0(x) = (2\pi)^{-n} \int_{\boldsymbol{R}^n} \Big\{ \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x)\xi^{\alpha+\beta} + 1 \Big\}^{-1} d\xi$$

and τ *is an arbitrary positive number smaller than* $(2m - an)/(6m - 2an - a)$.

2. Outline of the proof of the theorem.

Lemma 1. *There is a constant* K_3 *such that for any* $u \in W_{m,a}(\Omega)$ *and* $x \in \Omega$

$$|u(x)| \leq K_3 \delta(x)^{-an/4m} \|u\|_{m,a}^{n/2m} \|u\|_0^{1-n/2m}$$

Using Lemma 1 and the argument of Lemma 3.2 in our previous paper [2] we get the following Lemma.

Lemma 2. *Let* S *be a bounded operator on* V^* *to* V. *Then* S *has a kernel* M *in the following sense*

$$(Sf)(x) = \int_\Omega M(x, y) f(y) dy \qquad \text{for } f \in L^2(\Omega)$$

$M(x, y)$ *is continuous in* $\Omega \times \Omega$ *and there exists a constant* K_4 *such that*

$$|M(x, y)| \leq K_4 \delta(x)^{-an/4m} \delta(y)^{-an/4m} \|S\|_{V^* \to V}^{n^2/4m^2} \|S\|_{V^* \to L^2}^{n/2m - n^2/4m^2} \|S\|_{L^2 \to V}^{n/2m - n^2/4m^2} \|S\|_{L^2 \to L^2}^{(1-n/2m)^2}$$

for any $x, y \in \Omega$. *Here* $\|S\|_{V^* \to V}$ *denote the norm of* S *considered as an operater on* V^* *to* V *and similarly for other norms.*

For a complex number λ let $d(\lambda)$ be the distance from λ to the positive real axis. We set

$$a_{\alpha\beta}^1(x) = \begin{cases} a_{\alpha\beta}^0(x) & \text{if } |x - x_0| \leq \delta(x_0)/N \\ a_{\alpha\beta}^0(x_1) & \text{if } |x - x_0| > \delta(x_0)/N \end{cases}$$

where $a_{\alpha\beta}^0(x) = \sum_{|\gamma| \leq l} (x - x_0)^\gamma / \gamma!\, \partial_x^\gamma a_{\alpha\beta}(x_0)$, x_1 is the point of intersection of the sphere $|x - x_0| = \delta(x_0)/N$ and the line segment connecting x_0 and x, and l and N are sufficiently large integers. Next we put

$$a_{\alpha\beta}^2(x) = \rho_{\varepsilon_1} * a_{\alpha\beta}^1(x)$$

where $\varepsilon_1 = \delta(x_0)/2N$ and $\rho_\varepsilon *$ is the Friedrichs mollifier. Then we consider the differential operator with coefficients defined in $\boldsymbol{R}^n$: $P(x, D_x) = \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}^2(x) D^{\alpha+\beta}$.

Lemma 3. *If* $|\lambda|$ *is sufficiently large and* $\delta(x)^{-1+a/2m}|\lambda|^{-1/2m+\eta}(|\lambda|/d(\lambda))^2 \leq 1$ *for some small positive number* η, *then* $P-\lambda$ *has an inverse* $(P-\lambda)^{-1}$ *which maps* $L^2(\boldsymbol{R}^n)$ *into* $L^2(\boldsymbol{R}^n)$. *This operator has a kernel function* $K_\lambda^1(x, y)$ *such that for some constant* K_5

$$|K_\lambda^1(x, x) - C(x)(-\lambda)^{-1+n/2m}| \leq K_5 \delta(x)^{-an/2m+(a-2m)/2m} |\lambda|^{-1+(n-1)/2m}.$$

For the proof of Lemma 3 we use the method of M. Nagase [4] and the formula of the parametrix due to M. Nagase and K. Shinkai [5].

Lemma 4. *There exists a constant* K_6 *such that*

i) $\|(A-\lambda)^{-1}\|_{V^* \to V} \leq K_6 |\lambda| / d(\lambda)$

ii) $\|(A-\lambda)^{-1}\|_{V^* \to L^2} \leq K_6 |\lambda|^{1/2} / d(\lambda)$

iii) $\|(A-\lambda)^{-1}\|_{L^2 \to V} \leq K_6 |\lambda|^{1/2} / d(\lambda)$

iv) $\|(A-\lambda)^{-1}\|_{L^2 \to L^2} \leq K_6 / d(\lambda)$

if $d(\lambda) \geq K_7 |\lambda|^{1-1/2m}$ *and* $|\lambda| \geq K_7$, *here* K_7 *is a sufficiently large number.*

Following the method of our previous paper [2] we consider various operators and estimate their kernel functions. We denote by $K_\lambda(x, y)$ the kernel function of the operator $(A-\lambda)^{-1}$.

Lemma 5. *For any given number* η *we get the following estimate*:

$$\int_\Omega K_\lambda(x, x) dx = \int_\Omega C(x) dx \, (-\lambda)^{-1+n/2m} + O[(|\lambda|/d(\lambda))^{(6m-2an-a)/(2m-a)} \cdot |\lambda|^{-(2m-an)/2m(2m-a)+\eta}]$$

where $d(\lambda) \geq |\lambda|^{1-1/4m+\eta/2}$ *and* $|\lambda|$ *sufficiently large.*

Sketch of the proof. From Lemma 2 and Lemma 4 we get the inequality

$$|K_\lambda(x, x)| \leq K_8 \, \delta(x)^{-an/2m} |\lambda|^{n/2m} / d(\lambda).$$

On the other hand we get

$$\begin{aligned} \int_\Omega K_\lambda(x, x) dx &= \int_{\Omega - \Omega_\mu} (K_\lambda(x, x) - C(x)(-\lambda)^{-1+n/2m}) dx \\ &+ \int_{\Omega_\mu} (K_\lambda(x, x) - C(x)(-\lambda)^{-1+n/2m}) dx \\ &+ \int_\Omega C(x)(-\lambda)^{-1+n/2m} \, dx = I_1 + I_2 + I_3 \end{aligned}$$

where $\Omega_\mu = \{x \in \Omega \,;\, \delta(x) \geq \mu\}$.

We put $\mu = \{(|\lambda|/d(\lambda))^2 |\lambda|^{-1/2m+\eta/2}\}^{2m/(2m-a)}$. Then we know

$$|I_1| \leq K_9 (|\lambda|/d(\lambda))^{(6m-2an-a)/(2m-a)} |\lambda|^{-(2m-an)/2m(2m-a)+\eta}.$$

Using Lemma 3 and the estimates of the kernel functions $K_\lambda^1(x, x)$ and $K_\lambda(x, x)$ we find

$$|I_2| \leq K_{10} (|\lambda|/d(\lambda))^{(4m-2an)/(2m-a)} |\lambda|^{-(2m-an)/2m(2m-a)+\eta}.$$

Combining the above lemmas and following Tauberian argument of [3] we obtain the Theorem.

References

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