133. *Note on Singular Perturbation of Linear Operators*

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0. Introduction. Consider the following problem in a Banach space X:

$$\begin{cases} \partial u(t,\varepsilon)/\partial t + A(\varepsilon)u(t,\varepsilon)=0, & t>0, \\ u(0,\varepsilon)=a. \end{cases} \tag{0.1}$$

Here ε is a positive parameter, $0<\varepsilon\leqq 1$, $A(\varepsilon)=\varepsilon A+B$, and $a\in X$. We assume that A and B are closed linear operators in X with $\mathbf{D}(A)\subset\mathbf{D}(B)$ and that $-A(\varepsilon)$ with $\mathbf{D}(A(\varepsilon))=\mathbf{D}(A)$ generates a strongly continuous semi-group of bounded operators in X (i.e., of class (C_0)), uniformly with respect to ε; that is, with a constant $M>0$,

$$\|\exp(-tA(\varepsilon))\|\leqq M \tag{0.2}$$

for all $t\geqq 0$ and $0<\varepsilon\leqq 1$.

The (mild) solution of (0.1) is given by

$$u(t,\varepsilon)=\exp(-tA(\varepsilon))a, \qquad t\geqq 0,\ a\in X. \tag{0.3}$$

The map $]0,1]\in\varepsilon\mapsto u(t,\varepsilon)\in X$ is strongly continuous as seen immediately from the Trotter-Kato theorem (see Yosida [3], Kato [2]). However, $u(t,\varepsilon)$ may not be convergent as $\varepsilon\to 0$.

In the present note, we discuss a sufficient condition for the convergence of $u(t,\varepsilon)$ as $\varepsilon\to 0$. For that purpose, we introduce the set $C(p,\theta)$, $p>1$, $\theta<p-1$. $C(p,\theta)$ consists of all such elements b in $\mathbf{D}(A)$ that

$$\int_0^1 \varepsilon^\theta \sup_{t\geqq 0}\|A\exp(-tA(\varepsilon))b\|^q\, d\varepsilon<\infty. \tag{0.4}$$

It is easy to see that $C(p,\theta')\subset C(p,\theta)$ if $\theta'\leqq\theta$ and $C(q,\theta')\subset C(p,\theta)$ if $q\geqq p$, $p\theta'\leqq q\theta$.

Then we obtain the following

Theorem. *Let $b\in C(p,\theta)$ for some p,θ, $1<p<\infty$, $\theta<p-1$. Then* $\exp(-tA(\varepsilon))b$ *converges strongly to an element $b(t)\in X$ as $\varepsilon\to 0$, uniformly with respect to t in every compact interval. Furthermore,*

$$\exp(-tA(\varepsilon))b=b(t)+\boldsymbol{O}(\varepsilon^\rho), \qquad 0<\rho<1-\theta/(p-1). \tag{0.5}$$

Here $\boldsymbol{O}(\varepsilon^\rho)$ denotes the element in X such that $\varepsilon^{-\rho}\boldsymbol{O}(\varepsilon^\rho)$ remains bounded as $\varepsilon\to 0$, uniformly in t in every compact interval.

Let $\boldsymbol{D}=\bigcup_{p>1}\bigcup_{\theta<p-1}C(p,\theta)$. Then we immediately have

Corollary. *Let $\boldsymbol{D}$ be dense in X. Then there is an extension B_1*

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of B with the following two properties:

(i) $-B_1$ *generates a strongly continuous semi-group* $\exp(-tB_1)$;

(ii) $\exp(-tA(\varepsilon))a \to \exp(-tB_1)a$ *strongly as* $\varepsilon \to 0$ *for every* $a \in X$.

The convergence is uniform with respect to t in every compact interval.

Theorem will be proved by an elementary application of imbedding theorems for X-valued functions. In fact, this is a kind of the trace theorem. It seems that the requirement (0.4) is quite strong. In a certain sense, this is related to uniqueness property of the solution of (0.1) in a larger space, as will be seen in our proof. Our Corollary remains thus quite formal, for the substantial problem is, for instance, to determine when the set $\boldsymbol{D}$ is dense in X. On the other hand, if X is a Hilbert space, several results are known by using quadratic forms (see Kato [2], D. Huet (see the reference in [2]), Greenlee [1], etc).

1. Proof of Theorem. Let $b \in C(p, \theta)$, $1<p<\infty$, $\theta<p-1$. Put $s=(p-1)/(p-1-\theta)$. Then $s>0$. Now consider the following problem in $L^p(X)$:

$$(1.1) \qquad \begin{cases} \partial u(t, y)/\partial t + A(|y|^s)u(t, y) = 0, & t>0,\ |y|<1, \\ u(0, y) = b. \end{cases}$$

Here $L^p(X)$ is X-valued L^p-space over $|y|<1$. By our assumption, $-A(\varepsilon)$ generates a strongly continuous semi-group in X. Hence, an explicit construction of resolvents shows that $-A(|y|^s)$ generates a strongly continuous semi-group in $L^p(X)$. Here $\mathbf{D}(A(|y|^s))=\{f \in L^p(X)$; $f(y) \in D(A(|y|^s))$ *considered as element of X for a.e.y, and* $A(|y|^s)f(y) \in L^p(X)\}$. In particular, we have

$$u(t, y) = \exp(-tA(|y|^s))b.$$

Now consider the following system:

$$(1.2) \qquad \begin{cases} \partial v(t, y)/\partial t + A(|y|^s)v(t, y) = -sy|y|^{s-2} \quad Au(t, y), \ t>0, \\ v(0, y) = 0. \end{cases}$$

Note that the distribution derivative $\partial u(t, y)/\partial y$ formally satisfies (1.2). We prove in fact that $v(t, y) = \partial u(t, y)/\partial y$ under our hypothesis. (0.4) and the Lebesgue-Fatou convergence theorem imply that $y|y|^{s-2}Au(t, y) \in L^p(X)$ and is strongly continuous in t. Thus $v(t, y)$ as a mild solution of (1.2) is in $L^p(X)$. Let $\varphi=((\lambda+A(|y|^s))^{-1})^*\psi$, $\psi \in L^{p'}(X^*)$, $p'=p/(p-1)$. Here $*$ denotes the adjoint and $\operatorname{Re}\lambda \geqq \delta > 0$. We denote by $v^\wedge(\lambda, y)$ the Laplace transform of $v(t, y)$. Since for (almost) every y,

$$\int_0^\infty e^{-t\lambda} y|y|^{s-2}Au(t, y)dt = y|y|^{s-2}A(\lambda+A(|y|^s))^{-1}b,$$

(0.4) implies that

$$(1.3) \qquad y|y|^{s-2}A(\lambda+A(|y|^s))^{-1}b \in L^p(X).$$

Hence, noting that $((\bar{\lambda}+A(|y|^s)^*)^{-1} = ((\lambda+A(|y|^s))^{-1})^*$, we obtain

$$\langle v^\wedge(\lambda, y), \psi\rangle = -s\langle y|y|^{s-2}A(\lambda+A(|y|^s))^{-1}b, ((\lambda+A(|y|^s))^{-1})^*\psi\rangle.$$

Here $\langle\ ,\ \rangle$ denotes the coupling of $L^p(X)$ and $L^{p'}(X^*)$. It follows that

$$v^\wedge(\lambda, y) = -sy|y|^{s-2}(\lambda + A(|y|^s))^{-1}A(\lambda + A(|y|^s))^{-1}b.$$

Let χ be any X^*-valued differentiable function with compact support. Then

$$\langle v^\wedge(\lambda, y), \chi(y)\rangle = -\langle \lambda + A(|y|^s))^{-1}b, \chi'(y)\rangle.$$

In particular, $v^\wedge(\lambda, y)$ is holomorphic in $\mathrm{Re}\,\lambda > 0$. Now the inverse Laplace transform shows

$$\langle v(t, y), \chi(y)\rangle = -\langle u(t, y), \chi'(y)\rangle,$$

or $v(t, y) = \partial u(t, y)/\partial y$. We thus see $u(t, y) \in W^{p,1}(X)$, the X-valued L^p-Sobolev space. We then apply the imbedding theorem and see that $u(t, y)$ is Hölder continuous in y with exponent $\sigma < 1 - 1/p$. Rewriting this fact, taking $\varepsilon = |y|^s$, we obtain Theorem. Q.E.D.

2. Some discussions. It is immediately seen that $b \in C(p, \theta)$ is equivalent to:

$$(2.1) \quad \int_0^1 \varepsilon^{\theta-p} \sup_{t\geqq 0} \|\{\exp(-tA(\varepsilon))B - B\exp(-tA(\varepsilon))\}b\|^p \quad d\varepsilon < \infty.$$

In any case, this type of condition is difficult to verify. However, since the requirement of such a strong condition intervenes only for the proof of $v = \partial u/\partial y$, we can much relax the condition (0.4) or (2.1) in practical cases.

That $\mathbf{D}$ is dense in X, as required in our Corollary, is also quite strong, as suggested by the following consideration of the convergence of resolvents. Define $C^*(p, \theta; \lambda)$, $1 < p < \infty$, $\theta < p - 1$, $\mathrm{Re}\,\lambda > 0$, as the set of all $b \in X$ such that

$$(2.2) \qquad \int_0^1 \varepsilon^\theta \|(\lambda + A(\varepsilon))^{-1}A(\lambda + A(\varepsilon))^{-1}b\|^p \quad d\varepsilon < \infty.$$

Then we have

Proposition. *If $b \in C^*(p, \theta; \lambda)$, then $(\lambda + A(\varepsilon))^{-1}b$ converges strongly to an element $b(\lambda) \in X$ as $\varepsilon \to 0$, and*

$$(\lambda + A(\varepsilon))^{-1}b = b(\lambda) + O(\varepsilon^\rho), \qquad 0 < \rho < 1 - \theta/(p-1).$$

Note that $C(p, \theta) \subset C^*(p, \theta; \lambda)$ for all λ, $\mathrm{Re}\,\lambda > 0$ (see (1.3)). On the other hand, only with some additional uniformity requirement, $b \in C^*(p, \theta; \lambda)$ for all $\mathrm{Re}\,\lambda > 0$ implies $b \in C(p, \theta)$. Now put

$$\mathbf{D}^* = \bigcup_{p>1} \bigcup_{p-1>\theta} \bigcup_\lambda C^*(p, \theta; \lambda).$$

Thus the requirement that $\mathbf{D}^*$ be dense in X is apparently much weaker than that of denseness of $\mathbf{D}$. However, we then need a condition which assures that the pseudoresolvents obtained as the limits of $(\lambda + A(\varepsilon))^{-1}$ are in fact resolvents. By the way, we note that for the validity of Proposition, it is sufficient to require that $\|(\lambda + A(\varepsilon))^{-1}\| \leqq M(\lambda)$ uniformly with respect to ε with some function $M(\lambda)$.

References

[1] Greenlee, W. M.: Rate of convergence in singular perturbations. Ann. Inst. Fourier, **18**, 135–191 (1968).

[2] Kato, T.: Perturbation Theory for Linear Operators. Springer, Berlin (1966).

[3] Yosida, K.: Functional Analysis. Springer, Berlin (1965).