150. On Closed Graph Theorem

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(Comm. by Kinjirô KUNUGI, M. J. A., Nov. 13, 1972)

The closed graph theorem has been proved in some linear topological space. In this note we show that this theorem is true in a ranked space with some conditions. The theory of ranked space has been investigated by K. Kunugi since 1954. Throughout this note, $g, f, \cdots$ will denote points of a ranked space, $U_i, V_i, \cdots$ neighbourhoods of the origin with rank i, $\{U_{r_i}\}, \{V_{r_i}\}, \cdots$ fundamental sequences of neighbourhoods with respect to the origin. Let a linear space E be a complete ranked space with indicater ω_0, which satisfies the following conditions.

(E, 1) (1) For any neighbourhood U_i, the origin belongs to U_i.
(2) The E is the neighbourhood of the origin with rank zero.

(E, 2) Let U_i be any neighbourhood of the origin, λ be any number with $\lambda>0$ and g be a point in λU_i. If $\{V_{r_i}\}$ is a fundamental sequence of neighbourhoods, there is an integer i_0 such that $\lambda U_i \supset g+V_{r_j}$ for $j \geqq i_0$.

(E, 3) The following conditions are the modification of Washihara's conditions [4].

(R, L$_1$) For any $\{U_i\}$ and $\{V_i\}$, there is a $\{W_i\}$ such that $U_i+V_i \subseteqq W_i$.

(R, L$_2$)′ (1) For any $\{U_i\}$ and $\lambda>0$, there is a $\{V_i\}$ such that $\lambda U_i \subseteqq V_i$.
(2) For any $\{U_i\}$ and $\{\lambda_i\}$ with $\lim \lambda_i=0$, $\lambda_i>0$, there is a $\{V_i\}$ such that $\lambda_i U_i \subseteqq V_i$.

(R, L$_3$) Let g be any point in E. For any $\{U_i\}$ there is a $\{V_i(g)\}$, which is a fundamental sequence of neighbourhoods with respect to g, such that $g+U_i \subseteqq V_i(g)$ and conversely, for any $\{U_i(g)\}$ there is a $\{V_i\}$ such that $U_i(g) \subseteqq g+V_i$.

(E, 4) Let M be an absolutely convex set in E and V_i be a neighbourhood of the origin. If $\bar{M}$[1] $\supset f+V_i$, there is a $\lambda>0$ such that $\bar{M} \supset \lambda V_i$.

(E, 5) For given distinct points g_1, g_2, there exists some neighbourhood of the origin U_i such that $(g_1+U_i) \not\ni g_2$.

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1) $g \in \bar{M}$ if and only if there exists some sequence $\{g_i\}$ in M such that $g_i \to g$ in the sense of ranked space.

Next, let a linear space F be a ranked space with indicater ω_0, which satisfies the following conditions.

(F, 1)
(1) This is the same as (E, 1) (1).
(2) For any U_i and V_j, there is a W_k such that $W_k \subseteqq U_i \cap V_j$.
(3) For any U_i and integer n, there is an m such that $m \geqq n$, and a V_m such that $V_m \subseteqq U_i$.
(4) The F is the neighbourhood of the origin with rank zero.

(F, 2) The F satisfies the Washihara's conditions (R, L_1), (R, L_2) and (R, L_3), [4].

(F, 3) All neighbourhoods in F are absolutely convex. Moreover, some countable union of neighbourhoods of the origin with rank one absorbs all elements of F. In general, for any neighbourhood U_i, all elements of U_i are absorbed by some countable union of neighbourhoods of the origin with rank $(i+1)$, whose members are included in U_i.

(F, 4) The F is an R-complete space, that is, for any R-Cauchy sequence $\{g_i\}$, there is an element g such that $g_i \xrightarrow{R} g$ in F. (We say that the sequence $\{g_i\}$ in F is an R-Cauchy sequence, if there exists some fundamental sequence of neighbourhoods $\{V_{\gamma_i}\}$ such that $g_i - g_j \in V_{\gamma_i}$ for $j \geqq i$.)

(F, 5) This is the same as (E, 5).

Now, for convenience's sake we call F type a linear ranked space which satisfies the conditions (F, 1)$\sim$(F, 5). We have already understood that an R-complete ranked space has the following properties:

(1) The $\varphi(R)$ of an R-complete linear ranked space R by a continuous linear mapping φ is also an R-complete linear ranked space.

(2) The closed[2)] subspace of an R-complete linear ranked space is also an R-complete linear ranked space.

(3) The quotient space R/L, where R is an R-complete linear ranked space and L is a closed subspace, is an R-complete linear ranked space.

(4) The product space $\prod_{n=1}^{\infty} R_n$ of R-complete linear ranked spaces R_n $(n=1, 2, \cdots)$ is an R-complete linear ranked spaces.

(5) The inductive limite of R-complete linear ranked spaces R_n $(n=1, 2, \cdots)$ is an R-complete linear ranked space.

Moreover we can see easily that F type has also the properties above (1)$\sim$(5). Suppose $\{M_i\}$ is the family of sets in E and U is a neighbourhood in E, then if $\bigcup_{i=1}^{\infty} \overline{M}_i \supset U$, there exists some M_i such that $\overline{M}_i$ includes some neighbourhood in E. Now, we can prove the following theorem.

Theorem. *Let E, F be the above-mentioned space. And let T be*

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2) The set M is a closed set if $M = \overline{M}$.

a closed linear operator whose domain is all of E and whose range is in F. Then T is continuous.

Proof. Let $\mathfrak{M}_1$ be the countable family of the neighbourhoods of the origin with rank one, whose union absorbs all elements of F. Thus

$$\bigcup_{n=1}^{\infty} n\Big(\bigcup_{\mathfrak{M}_1 \ni U_1} T^{-1}(U_1)\Big)=E.$$

Then there exist some $n_0T^{-1}(U_1)$ and some neighbourhood $V^*_{\gamma_1}+f_1$ in E such that

$$\overline{n_0T^{-1}(U_1)}\supset V^*_{\gamma_1}+f_1.$$

Next, let $\mathfrak{M}_2$ be the countable family of the neighbourhoods of the origin with rank two, whose members are included in U_1, such that

$$\bigcup_{n=1}^{\infty} n\Big(\bigcup_{\mathfrak{M}_2 \ni U_2} U_2\Big)\supset U_1.$$

Then there exist some $n_0n_1T^{-1}(U_2)$ and some neighbourhood $V^*_{\gamma_2}+f_2$ in E such that

$$\overline{n_0n_1T^{-1}(U_2)}\supset V^*_{\gamma_2}+f_2.$$

In general, by induction we obtain

$$\overline{n_0\cdots n_{i-1}T^{-1}(U_i)}\supset V^*_{\gamma_i}+f_i, \qquad U_i\geqq U_{i+1}.$$

Consequently, by conditions (E, 2) and (E, 4), we can take

$$\overline{T^{-1}\Big(\frac{1}{2^{i+1}}\,U_i\Big)}\supset \mu V^*_{\gamma_i}$$

for all i, where $\{V^*_{\gamma_i}\}$ is a fundamental sequence of neighbourhoods with respect to the origin and $\{\mu_i\}$ is the sequence such that $\mu_i>0$ and $\mu_i\downarrow 0$.

Now, since T is the closed linear operator, we can prove that for all i

$$T^{-1}(U_i)\supset \overline{T^{-1}\Big(\frac{1}{2^{i+1}}\,U_i\Big)}\supset \mu_i V^*_{\gamma_i}.$$

Suppose $f_i\xrightarrow{R} f$ in E, that is, there exists some fundamental sequence $\{W^*_{\gamma_i}\}$ of neighbourhoods with respect to the origin such that $f_i-f\in W^*_{\gamma_i}$ for all i.

By (E, 2), for any $\mu_iV^*_{\gamma_i}$, there exists some integer $N=N(i)$ such that $\mu_iV^*_{\gamma_i}\supset W^*_{\gamma_j}$ to $j\geqq N(i)$. Thus, since

$$f_j-f\in W^*_{\gamma_j}\subset \mu_iV^*_{\gamma_i}\subset T^{-1}(U_i),$$

we have

$$Tf_j-Tf\in U_i \qquad \text{for } j\geqq N(i).$$

Hence,

$$Tf_i\xrightarrow{R} Tf \qquad \text{in } F.$$

Then we see that T is continuous.

We shall introduce new axiom.

(C') *Let U_i be any neighbourhood of the origin and $\{V_{\gamma_i}\}$ be any fundamental sequence of neighbourhoods with respect to the origin. If $g\in U_i$, there exists some integer $i_0\geqq 1$ such that $U_i\supset g+V_{\gamma_j}$ for $j\geqq i_0$.*

Let E' be the space E having axiom (C') in place of (E, 2). Then we have the following.

Corollary. *Let T be a closed linear operator whose domain is all of E' and whose range is in F. Then T is continuous.*

Finally, we note down that the class of F type includes, of course, the space $\mathfrak{D}'$ of L. Schwartz and the class of space E includes a non-metrizable space. The example of space E is the xy plane, whose base of the neighbourhood of the origin is $\{U_n\}$ $n=1, 2, \cdots$ such that

$$U_n=\left\{(x, y)\,;\, x^2+\left(y+\frac{1}{n}\right)^2<\frac{1}{n^2}\right\} \cup \{(0, 0)\}.$$

Acknowledgment. I wish to thank very much Prof. Kinjirô Kunugi for his valuable discussions and suggestions.

References

[1] K. Kunugi: Sur les espaces complets et régulièrement complets. I. Proc. Japan Acad., **30**, 553–556 (1954).
[2] ——: Sur la méthode des espaces rangés. I, II. Proc. Japan Acad., **42**, 318–322, 549–554 (1966).
[3] M. Washihara: On ranked spaces and linearity. Proc. Japan Acad., **43**, 584–589 (1967).
[4] ——: On ranked spaces and linearity. II. Proc. Japan Acad., **45**, 238–242 (1969).
[5] Y. Nagakura: The theory of nuclear spaces treated by the method of ranked space. I, II. Proc. Japan Acad., **47**, 337–341, 342–345 (1971).
[6] ——: The theory of nuclear spaces treated by the method of ranked space. III, IV. Proc. Japan Acad., **47**, 870–874, 875–879 (1971).