144. *On the Structure of Single Linear Pseudo-Differential Equations*

By Mikio SATO, Takahiro KAWAI, and Masaki KASHIWARA
Research Institute for Mathematical Sciences, Kyoto University
and Department of Mathematics, University of Nice

(Comm. by Kôsaku YOSIDA, M. J. A., Nov. 13, 1972)

The purpose of this note is to determine the structure of some class of *single* (linear) pseudo-differential equations by the aid of "quantized" contact transformations. (Cf. Egorov [1], Hörmander [4] and Sato, Kawai and Kashiwara [8].) It extends a result in § 2 of Chapter III of Sato, Kawai and Kashiwara [8] under the assumption of *single* equations.

Our main result is the following.

Theorem. *Let $P(x,D)$ be a pseudo-differential operator defined in a complex neighborhood U of $x_0^*=(x_0,\sqrt{-1}\,\eta_0)\in\sqrt{-1}\,S^*M$, where M is an n-dimensional real analytic manifold. Denote its principal symbol by $P_m(x,\eta)$. Assume that $P(x,D)$ satisfies conditions* (1) *and* (2) *below.*

Then the equation $P(x,D)u=0$ is micro-locally equivalent to one of the Mizohata equations

$$\mathfrak{M}_{k,l}^{\pm}:\left(\frac{\partial}{\partial x_1}\pm\sqrt{-1}\,x_1^k\frac{\partial}{\partial x_2}\right)^l u=0$$

considered near $(0;\sqrt{-1}\,(0,1,0,\cdots,0))$ *for some positive integers k and l.*

(1) *$V=\{(z,\zeta)\in U\,|\,P_m(z,\zeta)=0\}$ is a non-singular manifold.* (*Note that its defining ideal is not necessarily reduced.*)

(2) *There exist holomorphic functions $f_1(z,\zeta)$ and $f_2(z,\zeta)$ homogeneous in ζ such that $f_1=f_2=0$ on $V\cap\overline{V}$, $\overline{V}$ denoting the complex conjugate of V, and that their poisson bracket $\{f_1,f_2\}$ never vanishes.*

Proof. We denote by $Q(z,\zeta)$ a generator of the reduced defining ideal of V, i.e. $P_m=Q^l$. Then condition (2) assures that $d_{(z,\zeta)}Q(z,\zeta)$ and the canonical 1-form $\omega=\sum_{j=1}^n \zeta_j dz_j$ are linearly independent in a neighborhood of x_0^*. Hence by a suitable contact transformation we may assume without loss of generality that $Q(z,\zeta)$ has the form

$$\zeta_1+\sqrt{-1}\,\varphi(z,\zeta), \tag{3}$$

where $\varphi(z,\zeta)$ is real-valued on S^*M and that $V\cap\overline{V}=\{(x,\zeta)\,|\,z_1=0,\zeta_1=0\}$ (cf. Lemma 2.3.3 in Chapter III of Sato, Kawai and Kashiwara [8]). Then clearly $V\cap\overline{V}=\{\zeta_1=\varphi(z,\zeta)=0\}$. We can assume without loss of generality $(x_0,\eta_0)=(0;(0,1,0,\cdots,0))$. Therefore we can find an integer k so that $\varphi_0(z,\zeta')=\varphi(z,\zeta)|_{\zeta_1=0}$ has the form $\pm z_1^k\chi(z,\zeta')$ where χ never vanishes and is positive-valued at (x_0, η_0). Here and in the sequel ζ' denotes $(\zeta_2, \cdots, \zeta_n)$. Defining $\theta(z, \zeta)$ so that $\varphi(z, \zeta) = \varphi_0(z, \zeta') + \zeta_1 \theta(z, \zeta)$, $Q(z, \zeta)$ when multiplied by $1/(1+\sqrt{-1}\,\theta)$, acquires the form

$$q(z, \zeta) \pm \sqrt{-1}\, \tilde{r}(z, \zeta) \tag{4}$$

where

$$q(z, \zeta) = \zeta_1 \pm z_1^k \theta(z, \zeta) \chi(z, \zeta') / (1+\theta(z, \zeta)^2),$$
$$\tilde{r}(z, \zeta) = z_1^k \chi(z, \zeta') / (1+\theta(z, \zeta)^2).$$

Since $\chi(z, \zeta') / (1+\theta(z, \zeta)^2)$ never vanishes in a neighborhood of x_0^*, we may define a holomorphic function $r(z, \zeta)$ by

$$z_1 \sqrt[k]{\chi(z, \zeta') / (1+\theta(z, \zeta)^2)}.$$

Conditions (2) assures that

$$\{q(z, \zeta), r(z, \zeta)\} \neq 0 \tag{5}$$

in a neighborhood of x_0^*. By replacing r by $-r$ if necessary, we can assume that $\{q, r\} > 0$ holds on a real neighborhood of $(x_0, \eta_0) \in S^*M$.

Now we want to find a holomorphic function $a(z, \zeta)$ defined in a neighborhood of x_0^* (and homogeneous of degree $-1/k(k+1)$ in ζ) so that

$$\{a^k q, a r\} = 1 \tag{6}$$

and that

$$a(x_0^*) \neq 0. \tag{7}$$

Once such a function $a(z, \zeta)$ is obtained, we can apply the "quantized" contact transformation to $P(x, D)$ so that it takes the form $((D_1 + \sqrt{-1}\, z_1^k D_2) / \sqrt[k+1]{D_2})^l$ (cf. Theorem 5.3.7 in Chapter II of Sato, Kawai and Kashiwara [8]).

The existence of the required $a(z, \zeta)$ is proved in the following way:

If we can find a holomorphic function $A(z, \zeta; s, t)$ such that $A(x_0^*; 0, 0) \neq 0$ and that

$$\begin{aligned} &\frac{1}{k+1} t \frac{\partial A}{\partial t} + \frac{k}{k+1} s \frac{\partial A}{\partial s} + \frac{t}{k+1} \frac{\{q, A\}}{\{q, r\}} \\ &\quad + \frac{k}{k+1} s \frac{\{A, r\}}{\{q, r\}} + A = \frac{1}{\{q, r\}} \end{aligned} \tag{8}$$

holds, then $a(z, \zeta) = (A(z, \zeta; q(z, \zeta), r(z, \zeta)))^{1/(k+1)}$ clearly satisfies (6) and (7). Here the Poisson bracket $\{q, A\}$ (resp. $\{A, r\}$) means the Poisson bracket of q and A (resp. A and r) in which we regard s and t as irrelevant parameters. For simplicity of notations we define the derivations Λ_1 and Λ_2 in (z, ζ) by $\frac{1}{\{q, r\}} \{q, *\}$ and $\frac{-1}{\{q, r\}} \{r, *\}$ respectively.

Defining $B(z, \zeta; \lambda, s, t)$ by $\lambda^{k+1} A(z, \zeta; \lambda^k s, \lambda t)$, we can readily rewrite (8) in the following form:

$$\frac{1}{k+1} \frac{\partial}{\partial \lambda} B + \frac{1}{k+1} \Lambda_1(tB) + \frac{k}{k+1} \lambda^{k-1} \Lambda_2(sB) = \lambda^k / \{q(z, \zeta), r(z, \zeta)\}. \tag{9}$$

The hyperplane $\{\lambda = 0\}$ is clearly non-characteristic with respect to the first order differential equation (9), hence we can find a holomorphic solution $B(z, \zeta; \lambda, s, t)$ of (9) by giving 0 as its Cauchy datum on $\{\lambda=0\}$. Since neither A_1 or A_2 contains differentiation with respect to λ, the equation (9) clearly implies that

$$\frac{\partial^j}{\partial \lambda^j} B|_{\lambda=0}=0 \qquad \text{for } j=0, \cdots, k$$

and that

$$\frac{\partial^{k+1}}{\partial \lambda^{k+1}} B|_{\lambda=0}>0$$

in a real neighborhood Ω of $(z, \zeta, \lambda, s, t)=(x_0, \eta_0, 0, 0, 0)$. This implies that B/λ^{k+1} is holomorphic and positive-valued in Ω. Moreover, the expected homogeneity of B

$$B(c\lambda, c^{-k}s, c^{-1}t)=c^{k+1}B(\lambda, s, t)$$

is clearly satisfied, since B is the unique solution of the equation (9) with Cauchy datum 0.

This means that we can find $A(z, \zeta; s, t)$, whence also $a(z, \zeta)$ so that it satisfies (6) and (7). This completes the proof of the theorem.

Remark. The structure of the microfunction solution sheaf of the (pseudo-) differential equations

$$\mathfrak{M}_{k,l}^{\pm}: \left(\frac{\partial}{\partial x_1} \pm \sqrt{-1}\, x_1^k \frac{\partial}{\partial x_2}\right)^l u=0$$

is easily determined by the aid of the elementary solutions constructed in § 3.2 of Chapter I of Sato, Kawai and Kashiwara [8]. The result is as follows:

On a neighborhood Ω of $x_0^*=(0; \sqrt{-1}\,(0, 1, 0, \cdots, 0))$ we have for any l

(10) $\qquad \mathcal{E}xt_{\mathcal{P}}^j(\mathfrak{M}_{k,l}^{\pm}, C_M)=0 \qquad$ for any j, if k is even.

(Cf. Mizohata [6], Suzuki [10].)

(11) $$\mathcal{E}xt_{\mathcal{P}}^j(\mathfrak{M}_{k,l}^{+}, C_M)=\begin{cases} 0 & \text{for } j\neq 1 \\ C_N^l & \text{for } j=1 \quad \text{on } \{\eta_2>0\} \end{cases}$$

and

$$\mathcal{E}xt_{\mathcal{P}}^j(\mathfrak{M}_{k,l}^{-}, C_M)=\begin{cases} 0 & \text{for } j\neq 0 \\ C_N^l & \text{for } j=0, \end{cases} \quad \text{if } k \text{ is odd on } \{\eta_2>0\}$$

where C_N denotes the sheaf of microfunctions on $\sqrt{-1}\, S^*N$, where S^*N is identified with $\{(x, \eta) \in S^*M \mid x_1=\eta_1=0\}$.

Thus our theorem clearly extends the results on the (analytic) hypoellipticity and non-solvability of linear (pseudo-) differential equations obtained by many authors (sometimes only for distribution solutions) at the generic points on the characteristic variety. (Cf. e.g. Egorov [2], [3], Nirenberg and Trèves [7], Treves [11], Kawai [5], Schapira [9] and Suzuki [10].)

References

[1] Egorv, Ju. V: On canonical transformations of pseudo-differential operators. Uspehi Mat. Nauk, **24**, 235–236 (1969).

[2] ——: On the condition of solvability of pseudo-differential equations. ential equations. I. Comm. Pure Appl. Math., **23**, 1–38 (1970).

[3] ——: On subelliptic pseudo-differential operators. Dokl. Acad. Nauk, **188**, 20–22 (1969).

[4] Hörmander, L.: Fourier integral operators. I. Acta Math., **127**, 79–183 (1971).

[5] Kawai, T.: Construction of local elementary solutions for linear differential operators. II. Publ. RIMS, Kyoto Univ., **7**, 399–426 (1971).

[6] Mizohata, S.: Solutions nulles et solutions non analytiques. J. Math. Kyoto Univ., **1**, 271–302 (1962).

[7] Nirenberg, L., and F. Treves: On local solvability of linear partial differential equations. I. Comm. Pure Appl. Math., **23**, 1–38 (1970).

[8] Sato, M., T. Kawai, and M. Kashiwara: Microfunctions and pseudo-differential equations (to appear in the report of Katata symposium).

[9] Schapira, P.: Solutions hyperfonctions des équations aux dérivées partielles du premier ordre. Bull. Soc. Math. France, **97**, 243–255 (1969).

[10] Suzuki, H.: Local existence and analyticity of hyperfunction solutions of partial differential equations of first order in two independent variables. J. Math. Soc. Japan, **23**, 18–26 (1971).

[11] Trèves, F.: Analytic-hypoelliptic partial differential equations of principal type. Comm. Pure Appl. Math., **24**, 537–570 (1971).