6. A Remark on Fluid Flows through Porous Media

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(Comm. by Kôsaku YOSIDA, M. J. A., Jan. 12, 1973)

1. Introduction. According to Muskat [10], the mathematical model for flow through a homogeneous porous medium is the following degenerate quasilinear parabolic equation

$$(*)\qquad \frac{\partial u}{\partial t}=\Delta u^m,$$

where u is the density distribution, Δ is the Laplace-Beltrami operator in the space variable x and m is a real constant >1. Physically $m-1$ is the ratio of specific heats c_p/c_v. Equations of this type are of great importance in technology (see Ames [1], 1.2); besides they have some properties which seem interesting from a purely mathematical point of view. See Oleinik *et al.* [11], § 4; the author would refer the reader to the recent elaborate studies by Aronson [2]-[4]. (Of course the equation $(*)$ has been studied by many other authors from various interesting aspects.)

To avoid unnecessary technical difficulties we concentrate our attention to flows through a medium which occupies all of the circle S^1. We consider the following Cauchy problem

$$(1)\qquad \begin{cases} \dfrac{\partial u}{\partial t}=\dfrac{\partial^2}{\partial x^2}u^m & \text{in } S^1\times(0,T),\\ u|_{t=0}=a(x) & x\in S^1,\end{cases}$$

where a is a given non-negative function on S^1 called a initial datum and $0<T<\infty$. If $a\in C(S^1)^+$ and $da^m/dx\in L^\infty(S^1)$, then the Cauchy problem (1) has a unique "weak solution" u such that $u\in C(S^1\times[0,T])^+$ and $\partial u^m/\partial x\in L^\infty(S^1\times(0,T))$ (cf. Oleinik *et al.* [11]). Here and throughout the paper we use the usual vector lattice notation, i.e., $C(S^1)^+$ is the cone of all non-negative elements of $C(S^1)$ etc. da^m/dx and $\partial u^m/\partial x$ are distribution derivatives of $a^m\in\mathcal{D}'(S^1)$ and $u^m\in\mathcal{D}'(S^1\times(0,T))$ respectively.

The purpose of the present paper is to show the *continuous dependence of weak solutions on the initial data* in the sense of $L^1(S^1)$. Our result reads:

Theorem. *Suppose that*

$$a,\hat a\in C(S^1)^+\quad \textit{and}\quad \frac{d}{dx}a^m,\frac{d}{dx}\hat a^m\in L^\infty(S^1).$$

Let u and $\hat u$ be the corresponding unique weak solutions of (1) *such that*

$$u, \hat{u} \in C(S^1 \times [0, T])^+ \quad \textit{and} \quad \frac{\partial}{\partial x} u^m, \frac{\partial}{\partial x} \hat{u}^m \in L^\infty(S^1 \times (0, T)).$$

Then we have

(2) $\|u(t) - \hat{u}(t)\|_{L^1(S^1)} \leqslant \|a - \hat{a}\|_{L^1(S^1)}$ *for* $0 \leqslant t \leqslant T$.

Moreover

(3) $\|(u(t) - \hat{u}(t))^+\|_{L^1(S^1)} \leqslant \|(a - \hat{a})^+\|_{L^1(S^1)}$ *for* $0 \leqslant t \leqslant T$.

The proof relies upon the recent *theory of nonlinear semi-groups* in general Banach spaces (see Crandall and Liggett [6]). The properties (2) and (3) have been already pointed out in Konishi [8] in a somewhat indirect form. See also Crandall [5] (Theorem 4.12), Konishi [7] (Theorem C), Konishi [9] (Lemma 1), Sato [12] (5.2), Vol'pert and Hudjaev [13] (Theorem 2.2). The following property becomes also clear by the proof of our Theorem: *Under the assumption of the Theorem, one has*

$$\|u(t)\|_{L^p(S^1)} \leqslant \|a\|_{L^p(S^1)} \qquad \textit{for } 0 \leqslant t \leqslant T$$

whenever $1 \leqslant p \leqslant \infty$ (cf. Theorem 4 of Konishi [8]).

2. Proof of the theorem. Let us write down the basic result obtained in the appendix II of Konishi [8]. We define a nonlinear operator A in the dual space $C(S^1)^*$ of $C(S^1)$:

$$D(A) = \{f \in C(S^1)\,;\, df|f|^{m-1}/dx \in L^\infty(S^1) \text{ and } d^2f|f|^{m-1}/dx^2 \in C(S^1)^*\},$$

$$Af = d^2f|f|^{m-1}/dx^2 \qquad \text{for } f \in D(A).$$

Note that the space $C(S^1)^*$ is the Banach space of bounded Baire measures on S^1 with the norm of total variation (Riesz-Markov-Kakutani theorem) and it contains $L^1(S^1)$ as a closed subspace. We know that the operator A is dissipative in $C(S^1)^*$, i.e.,

$$\|f - g - \lambda(Af - Ag)\|_{C(S^1)^*} \geqslant \|f - g\|_{C(S^1)^*}$$

whenever $\lambda > 0$ for $f, g \in D(A)$ and it satisfies

$$R(I - \lambda A) \supset D(A) \qquad \text{for } \lambda > 0.$$

Hence A generates a nonlinear contraction semi-group $\{e^{tA}\}_{t \geqslant 0}$ on $\overline{D(A)} = L^1(S^1) \subset C(S^1)^*$ in the sense of Theorem I of Crandall and Liggett [6]:

$$e^{tA}: L^1(S^1) \to L^1(S^1), \qquad t \geqslant 0;$$

$$e^{tA}f = \operatorname*{s-lim}_{n \to \infty} \left(I - \frac{t}{n}A\right)^{-n} f \quad \text{in } L^1(S^1), \quad f \in D(A), \quad t \geqslant 0;$$

$$e^{tA}e^{sA} = e^{(t+s)A}, \quad t, s \geqslant 0;$$

$$e^{0A} = I;$$

$$\operatorname*{s-lim}_{t \downarrow 0} e^{tA}f = f \quad \text{in } L^1(S^1), \quad f \in L^1(S^1);$$

$$\|e^{tA}f - e^{tA}g\|_{L^1(S^1)} \leqslant \|f - g\|_{L^1(S^1)}, \quad f, g \in L^1(S^1), \quad t \geqslant 0.$$

Moreover $\{e^{tA}\}_{t \geqslant 0}$ is an *order-preserving semi-group*:

$$\|(e^{tA}f - e^{tA}g)^+\|_{L^1(S^1)} \leqslant \|(f - g)^+\|_{L^1(S^1)}, \quad f, g \in L^1(S^1), \quad t \geqslant 0.$$

As for the differentiability of $\{e^{tA}\}_{t \geqslant 0}$, we have the following: If $a \in D(A)$, then $u(t) = e^{tA}a \in D(A)$ for each $t \geqslant 0$, $u(t)$ is weakly* continuously differentiable in $C(S^1)^*$ in $t \geqslant 0$ and

$$\text{w}^*\text{-}\frac{d}{dt}u(t)=Au(t) \quad \text{in } C(S^1)^* \quad \text{for } t \geqslant 0.$$

More precisely, for a non-negative $a \in D(A)$, we obtain the following

Proposition (Konishi [8]). *Assume that*

$$a \in C(S^1)^+, \quad \frac{d}{dx}a^m \in L^\infty(S^1) \quad \textit{and} \quad \frac{d^2}{dx^2}a^m \in C(S^1)^*.$$

Then there exists a unique solution u of the problem (1) *such that*

$$u \in C(S^1 \times [0, T])^+,$$

$$\frac{\partial}{\partial x}u^m \in L^\infty(S^1 \times (0, T)),$$

$$\frac{\partial^2}{\partial x^2}u(t)^m \in C(S^1)^* \quad \textit{for } 0 \leqslant t \leqslant T.$$

Moreover

$$\left\|\frac{\partial^2}{\partial x^2}u(t)^m\right\|_{C(S^1)^*} \leqslant \left\|\frac{d^2}{dx^2}a^m\right\|_{C(S^1)^*}, \quad 0 \leqslant t \leqslant T,$$

and

$$\|u(t)\|_{L^p(S^1)} \leqslant \|a\|_{L^p(S^1)} \quad (1 \leqslant p \leqslant \infty), \quad 0 \leqslant t \leqslant T.$$

This solution u is expressed by

$$u(t)=e^{tA}a, \qquad 0 \leqslant t \leqslant T.$$

Our proof of the theorem is completed if we examine the following

Lemma. *Under the assumption of the theorem we have the expression:*

$$u(t)=e^{tA}a \quad \textit{and} \quad \hat{u}(t)=e^{tA}\hat{a} \quad \textit{for } 0 \leqslant t \leqslant T.$$

Proof. Let $\{a_n\}_{n \geqslant 1} \subset \mathcal{D}(S^1)$ be a monotone non-increasing sequence of strictly positive functions on S^1 such that

$$\sup_{n \geqslant 1}\left\|\frac{d}{dx}a_n^m\right\|_{L^\infty(S^1)} < \infty$$

and that

$$\lim_{n \to \infty} a_n(x)=a(x) \quad \text{at each} \quad x \in S^1.$$

We denote by u_n $(n \geqslant 1)$ the classical solutions of (1) with the initial data a_n; note that such u_n's surely exist. We can write that

$$u_n(t)=e^{tA}a_n, \quad n \geqslant 1, \quad 0 \leqslant t \leqslant T.$$

We obtain

$$0 \leqslant \cdots \leqslant u_n(t) \leqslant \cdots \leqslant u_2(t) \leqslant u_1(t) \quad \text{in } S^1, \quad 0 \leqslant t \leqslant T,$$

and the weak solution u can be obtained through

$$u(x, t)=\lim_{n \to \infty} u_n(x, t) \quad (x, t) \in S^1 \times [0, T]$$

(cf. Oleinik *et al.* [11]). Hence

$$u(t)=\text{s-}\lim_{n \to \infty} e^{tA}a_n \quad \text{in } L^1(S^1) \quad \text{for } 0 \leqslant t \leqslant T.$$

On the other hand,

$$a=\text{s-}\lim_{n \to \infty} a_n \qquad \text{in } L^1(S^1).$$

Accordingly

$$u(t)=e^{tA}a, \qquad 0\leqslant t\leqslant T.$$

Similarly

$$\hat{u}(t)=e^{tA}\hat{a}, \qquad 0\leqslant t\leqslant T. \qquad \text{Q.E.D.}$$

Remark. Set

$$\mathcal{M}=\left\{f\in C(S^1)^+\,;\,\frac{d}{dx}f^m\in L^\infty(S^1)\right\}.$$

Then

$$L^1(S^1)^+\cap D(A)\subset\mathcal{M}\subset L^p(S^1)^+ \qquad (1\leqslant p\leqslant\infty)$$

and

$$e^{tA}(L^1(S^1)^+\cap D(A))\subset L^1(S^1)^+\cap D(A),$$
$$e^{tA}\mathcal{M}\subset\mathcal{M},$$
$$e^{tA}L^p(S^1)^+\subset L^p(S^1)^+ \qquad (1\leqslant p\leqslant\infty)$$

for each $t\geqslant 0$. Thus we are now able to grasp the weak solutions of (1) (in the sense of Oleinik *et al.* [11]) within the scope of the theory established by Crandall and Liggett [6]. We hope that this result may stimulate us to further operator theoretical studies of the equation $(*)$. Our approach may be generalized to several directions. The details will be studied elsewhere.

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