30. *A Note on a Problem of Matlis*

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Following Faith and Walker [2] a module is said to be completely decomposable if it is a direct sum of indecomposable injective submodules. And a right ideal I of a ring R is called irreducible if $I \neq R$ and $I = I_1 \cap I_2$ implies $I = I_1$ or $I = I_2$, for all right ideals I_1 and I_2 of R.

It is an open problem whether every direct summand of a completely decomposable module is also completely decomposable, and E. Matlis [5] proved that we have an affirmative answer for modules over a right Noetherian ring. Recently in [6] we have proved that if a ring is non-singular and satisfying the ascending chain condition for essential right ideals its answer is also in the affirmative. Further it is known by us that the non-singular condition of them can be removed. Thus, in this note, using a result of Harada and Sai [3], we shall prove it as a corollary to the theorem which is a special case, concerning the completely decomposable modules, of the Krull—Remak—Schmidt—Azumaya's theorem. Namely,

Theorem 1. *The following conditions are equivalent.*

(I) *A ring R satisfies the ascending chain condition for irreducible right ideals.*

(II) *A ring R satisfies the ascending chain condition for essential, irreducible right ideals.*

(III) *If a completely decomposable module M_R has two direct sum decompositions in which each component is indecomposable, injective submodule;*

$$M = \sum_{i \in I} \oplus M_i = \sum_{j \in J} \oplus N_j,$$

then for any subset $I' \subset I$ (resp. $J' \subset J$) *there exists a one-to-one mapping φ of I' into J* (resp. J' *into* I) *such that $M_i \cong N_{\varphi(i)}$ for all $i \in I'$* (resp. $N_j \cong M_{\varphi(j)}$ *for all* $j \in J'$) *and*

$$M = \sum_{i \in I'} \oplus N_{\varphi(i)} \oplus \sum_{i \in I - I'} \oplus M_i$$

$$\left(\text{resp. } M = \sum_{j \in J'} \oplus N_j \oplus \sum_{i \in I - \varphi(J')} \oplus M_i\right).$$

Corollary. *If a ring satisfies the equivalent condition in Theorem 1, then every direct summand of a completely decomposable module is also completely decomposable.*

In case a ring R is right Noetherian the theorem is a part of [3;

Proposition 10—Corollary]. However, as was seen in [6], a ring satisfying the condition (II) in Theorem 1 is not necessarily right Noetherian. Thus, Corollary is a generalization of a result of Matlis [5] who proved, as mentioned above, the case of a right Neotherian ring. It should be noted that not every ring satisfies the condition (II) (e.g. indiscrete valuation ring).

For the proof of Theorem 1 we use the following lemma of Harada and Sai [3].

Lemma. *For any completely decomposable module the condition* (III) *in Theorem* 1 *holds if and only if, for any family of indecomposable injective modules* $\{M_n \mid n \geqq 1\}$ *and non-isomorphisms* $\{f_n : M_n \to M_{n+1} \mid n \geqq 1\}$, *and for any element* $x \in M_1$, *there exists an integer n such that* $f_n f_{n-1} \cdots f_1(x) = 0$.

Moreover, in this case every direct summand of a completely decomposable module is completely decomposable.

Proof. Since an endomorphism ring of an indecomposable injective module is local, this lemma is a special case of [3; Theorem 9].

Proof of Theorem 1.

(I)$\Rightarrow$(II). Trivial.

(II)$\Rightarrow$(III). Assume that there exist a family of non-ismorphisms $\{f_n : M_n \to M_{n+1} \mid n \geqq 1, M_n$ is indecomposable injective$\}$ and a non-zero element $x \in M_1$ such that $f_n \cdots f_1(x) \neq 0$ for any $n \geqq 1$. Then, since each f_n isnot a monomorphism, $\operatorname{Ker} f_n \cdots f_1 \neq 0$ and $\operatorname{Ker} f_{n+1} f_n \cdots f_1 / \operatorname{Ker} f_n \cdots f_1$ is essential in $M_1 / \operatorname{Ker} f_n \cdots f_1$. For the last fact, it suffices to show that $\operatorname{Ker} f_{n+1} f_n \cdots f_1 / \operatorname{Ker} f_n \cdots f_1$ is not zero, because $M_1 / \operatorname{Ker} f_n \cdots f_1$ is isomorphic to a submodule $f_n \cdots f_1(M_1)$ of M_{n+1}, which is uniform. Since $\operatorname{Ker} f_{n+1} f_n \cdots f_1 = (f_n \cdots f_1)^{-1} (\operatorname{Ker} f_{n+1} \cap \operatorname{Im} f_n \cdots f_1)$, $\operatorname{Ker} f_{n+1} \cap \operatorname{Im} f_n \cdots f_1 \neq 0$ and $(f_n \cdots f_1)(\operatorname{Ker} f_{n+1} f_n \cdots f_1) = \operatorname{Ker} f_{n+1} \cap \operatorname{Im} f_n \cdots f_1 \neq 0$, if $\operatorname{Ker} f_{n+1} f_n \cdots f_1 = \operatorname{Ker} f_n \cdots f_1$ for some n, then $(f_n \cdots f_1)(\operatorname{Ker} f_{n+1} f_n \cdots f_1) = (f_n \cdots f_1)(\operatorname{Ker} f_n \cdots f_1) = 0$, which is a contradiction. Hence $(0 : f_n \cdots f_1(x)) \subsetneqq (0 : f_{n+1} f_n \cdots f_1(x))$ for each $n \geqq 1$, because, since $0 \neq \bar{x} \in M_1 / \operatorname{Ker} f_n \cdots f_1$, there exists $r \in R$ such that $0 \neq \bar{x} r \in \operatorname{Ker} f_{n+1} f_n \cdots f_1 / \operatorname{Ker} f_n \cdots f_1$. This shows that $f_{n+1} f_n \cdots f_1(x) r = 0$ and $f_n \cdots f_1(x) r \neq 0$, that is, $r \in (0 : f_{n+1} f_n \cdots f_1(x))$ and $r \notin (0 : f_n \cdots f_1(x))$.

Now, there exists a non-zero element $f_1(x) a \in f_1(x) R \cap \operatorname{Ker} f_2$ for some $a \in R$ since M_2 is uniform and f_2 is not a monomorphism. Putting $y = xa$, a right ideal $I = \{r \in R \mid xr \in yR\}$ is essential in R. Then, for any $r \in I$ $f_2 f_1(x) r = f_2 f_1(xr) \subset f_2 f_1(yR) \subset f_2(\operatorname{Ker} f_2)$ and $f_2(\operatorname{Ker} f_2) = 0$. Hence $I \subset (0 : f_2 f_1(x))$ and $(0 : f_n \cdots f_1(x))$ is therefore essential in R for $n \geqq 2$. On the other hand, since each M_n is uniform and $R / (0 : f_n \cdots f_1)$ is isomorphic to $f_n \cdots f_1(x) R$ which is a submodule of M_{n+1}, $R / (0 : f_n \cdots f_1(x))$ is uniform and hence $(0 : f_n \cdots f_1(x))$ is irreducible. Thus we have a strictly ascending chain of essential, irreducible right ideals $\{(0 : f_n \cdots$

$f_1(x)) \mid n=2\}$ which contradicts to the condition (II). And therefore we have the condition (III) by lemma.

(III)$\Rightarrow$(I). Assume that we have a strictly ascending chain of irreducible right ideals $\{I_n \mid n \geqq 1\}$. Then we can define a non-isomorphism $g_n: R/I_n \to R/I_{n+1}$ for each n by putting $g_n(r+I_n)=r+I_{n+1}$ for $r \in R$. Since R/I_n is uniform right module, the injective hull $E(R/I_n)$ is indecomposable. Hence, if we extend g_n to $f_n: E(R/I_n) \to E(R/I_{n+1})$, the family $\{f_n \mid n \geqq 1\}$ is of non-isomorphisms and $f_n \cdots f_1(1+I_1) \neq 0$ for any $n \geqq 1$. This contradicts the condition (III) by Lemma. q.e.d.

Now then, Corollary is immediately obtained from Theorem 1 and Lemma.

In [1], a direct sum decomposition $M=\sum_{i \in I} \oplus M_i$ of a module M is said to complement direct summands in case for each direct summand N of M there is a subset $J \subset I$ with $M=N \oplus \sum_{j \in J} \oplus M_j$. Then, applying this notion to completely decomposable modules, it is easy to see that each completely decomposable module has a decomposition that complements direct summands if and only if the equivalent condition in Lemma holds for any family of completely decomposable modules and non-isomorphisms $\{f_n: M_n \to M_{n+1} \mid n \geqq 1\}$, in view of [4; Corollary to Theorem 4] and [1; Remark]. Thus we can restate Theorem 1 as the following (c.f. [1; Theorem 8]).

Theorem 2. *A ring satisfies the ascending chain condition for essential, irreducible right ideals if and only if every completely decomposable module has a decomposition that complements direct summands.*

References

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