40. On G_δ-Sets in the Product of a Metric Space and a Compact Space. II

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In [2] we defined G_δ-*space* as a topological space which is homeomorphic to a G_δ-set in the product of a metric space and a compact Hausdorff space and proved that an M-space is a G_δ-space if and only if it is a p-space. We also left it as an open problem to give an internal characterization to G_δ-spaces. The purpose of this note is to give such characterizations. All spaces in this note are at least Hausdorff, and all maps are continuous. As for general terminologies and symbols in general topology, see [1].

Definition 1. Let $\{\mathcal{W}_i \mid i=1,2,\cdots\}$ be a sequence of open covers of a space X and $\mathcal{F}$ a filter in X. Then $\mathcal{F}$ is said to be *Cauchy w.r.t.* $\{\mathcal{W}_i\}$ provided for each i there is $F\in\mathcal{F}$ and $W\in\mathcal{W}_i$ with $F\subset W$. Suppose S is a closed set of X. If every maximal closed filter $\mathcal{F}$ in X which is Cauchy w.r.t. $\{\mathcal{W}_i\}$ and contains S as an element converges, then S is said to be *complete w.r.t.* $\{\mathcal{W}_i\}$. We may drop the word 'w.r.t. $\{\mathcal{W}_i\}$' discussing Cauchy filter or complete closed set if there is no fear of confusion.

Definition 2. Let X be a space with a sequence $\{\mathcal{W}_i\}$ of open covers and f a map from X onto a space Y. If for every $y\in Y$ and for every maximal closed filter $\mathcal{F}$ in X, Cauchy w.r.t. $\{\mathcal{W}_i\}$ satisfying $f^{-1}(y)\notin\mathcal{F}$, there is $G\in\mathcal{F}$ such that $y\notin\overline{f(G)}$, then f is said to be *closed w.r.t.* $\{\mathcal{W}_i\}$. Obviously each closed map from X onto Y is closed w.r.t. every sequence $\{W_i\}$ of open covers of X.

Theorem 1. *A Tychonoff space X is a G_δ-space (namely homeomorphic to a G_δ-set in the product of a metric space and a compact Hausdorff space) if and only if there is a sequence $\{\mathcal{W}_i \mid i=1,2,\cdots\}$ of open covers of X and a map f from X onto a metric space M such that*
(i) *for each $y\in M$, $f^{-1}(y)$ is complete w.r.t. $\{\mathcal{W}_i\}$,*
(ii) *f is closed w.r.t. $\{\mathcal{W}_i\}$.*

Proof. *Necessity.* Let X be a G_δ-set in the product space $C\times M$ of a compact Hausdorff space C and a metric space M. Suppose $X=\bigcap_{i=1}^\infty U_i$, where U_i is an open set of $C\times M$. We denote by π_1 and π_2 the projections from $C\times M$ onto C and M respectively. For each point

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$x=(u, y)$ of X and each natural number n we define a nbd (=neighborhood) $T_n(x)$ which is a product of an open nbd of u in C and an open nbd of y in M and satisfies $\overline{T_n(x)} \subset U_n$. Note that $\bar{T}_n$, $\bar{W}_n$ and $\bar{F}$ denote closures in $C \times M$ (but not in X) throughout this part of the proof. Put

$$\mathcal{W}_n=\{W_n(x) \mid x \in X\}, \quad \text{where} \quad W_n(x)=T_n(x) \cap X.$$

Then $\mathcal{W}_n$, $n=1, 2, \cdots$ are open covers of X. Let f be the restriction of π_2 to X; then we may assume without loss of generality that f is a map from X onto the metric space M. To prove (i) let $\mathcal{F}$ be a maximal closed Cauchy filter w.r.t. $\{\mathcal{W}_i\}$ such that $f^{-1}(y) \in \mathcal{F}$. Then for each n there is $F_n \in \mathcal{F}$ and $W_n \in \mathcal{W}_n$ such that $F_n \subset W_n \subset \bar{W}_n \subset \bar{T}_n \subset U_n$, and $F_n \subset f^{-1}(y)$. Since C is compact, $\cap\{\bar{F} \mid F \in \mathcal{F}\}=\{x\}$ for some $x \in C \times M$. On the other hand $x \in \bar{F}_n \subset \bar{W}_n \subset U_n$, $n=1, 2, \cdots$, and hence $x \in \bigcap_{n=1}^{\infty} U_n =X$ follows. Thus $\mathcal{F} \rightarrow x$ in X proving (i).

To prove (ii) assume that $\mathcal{F}$ is a maximal closed Cauchy filter such that $f^{-1}(y) \notin \mathcal{F}$ for a point y of M. Then $f^{-1}(y) \cap F=\emptyset$ holds for some $F \in \mathcal{F}$. Assume $f(G) \cap S_\varepsilon(y) \neq \emptyset$ for all $\varepsilon>0$ and for all $G \in \mathcal{F}$, where $S_\varepsilon(y)$ denotes the ε-nbd of y in M. Then $f^{-1}(\overline{S_\varepsilon(y)}) \in \mathcal{F}$ follows from the maximality of $\mathcal{F}$. The filter base $\pi_1(\mathcal{F})=\{\pi_1(F) \mid F \in \mathcal{F}\}$ converges to a point u_0 in C because C is compact. Recall that $\mathcal{F}$ is Cauchy, and hence there are $F_n \in \mathcal{F}$ and $W_n \in \mathcal{W}_n$ with $F_n \subset W_n \subset \bar{W}_n \subset \bar{T}_n$. If $\bar{T}_n \cap \pi_2^{-1}(y) =\emptyset$ for some n, then $f(F_n) \subset \overline{\pi_2(T_n)} \not\ni y$, i.e. $y \notin \overline{f(F_n)}$, which proves (ii). (Note that T_n is the product of an open set in C and an open set in M.) Hence we suppose $\bar{T}_n \cap \pi_2^{-1}(y) \neq \emptyset$, $n=1, 2, \cdots$. Then $x_0=(u_0, y) \in \bar{T}_n \cap \pi_2^{-1}(y) \subset U_n$, $n=1, 2, \cdots$. Hence $x_0 \in X$. Let U be a nbd of u_0 in C. Then since $\pi_1(\mathcal{F}) \rightarrow u_0$, $\pi_1^{-1}(\bar{U}) \cap X \in \mathcal{F}$. Thus $X \cap \pi_1^{-1}(\bar{U}) \cap f^{-1}(\overline{S_\varepsilon(y)}) \in \mathcal{F}$ for every $\varepsilon>0$. This proves that $\mathcal{F} \rightarrow x_0$ in X. Therefore $x_0=(u_0, y) \in F$ for every $F \in \mathcal{F}$, and accordingly $f^{-1}(y) \cap F \neq \emptyset$ follows. This is a contradiction. Therefore $f(G) \cap S_\varepsilon(y)=\emptyset$ for some $\varepsilon>0$ and $G \in \mathcal{F}$, which means $y \notin \overline{f(G)}$ in M proving that f is closed w.r.t. $\{\mathcal{W}_n\}$.

Sufficiency. Since X is Tychonoff, we may assume without loss of generality that each $\mathcal{W}_n$ consists of cozero open sets (see [2]). We define a map g from X into $\beta X \times M$ by $g(x)=(x, f(x))$, $x \in X$. Then f is obviously a topological map from X onto a subset X' of $\beta X \times M$. Now, all we have to show is that X' is G_δ in $\beta X \times M$. Let us recall some properties of βX. Each point of βX may be regarded as a maximal filter of zero sets in X. Let U be a cozero open set in X. Then we put $\tilde{U}=\{z \in \beta X \mid X-U \notin z\}=\{z \in \beta X \mid F \subset U \text{ for some } F \in \mathcal{F}\}$. These $\tilde{U}$ for the cozero open sets U in X form a base for βX. Now for each n, $W'_n=\cup\{\tilde{W} \mid W \in \mathcal{W}_n\}$ is obviously an open set of βX such that $W'_n \supset X$. Therefore $W=\bigcap_{n=1}^{\infty} W'_n$ is a G_δ-set in βX consisting of all maximal zero set filters in X which are Cauchy w.r.t. $\{\mathcal{W}_n\}$. Thus $W \times M$ is G_δ in $\beta X \times M$. We can express X' as

(1) $$X'=\{(z, y)\in W\times M \mid f^{-1}(y)\in z\}.$$
It is obvious that each $(z, y)\in X'$ satisfies $f^{-1}(y)\in z$ and $(z, y)\in W\times M$. Conversely, let $(z, y)\in W\times M$ satisfies $f^{-1}(y)\in z$. Then, since $z\in W$, there is a maximal closed filter $\mathcal{F}$ containing z as a subcollection. Therefore $\mathcal{F}$ is Cauchy w.r.t. $\{W_n\}$. Since $f^{-1}(y)\in\mathcal{F}$, $\mathcal{F}\rightarrow x\in f^{-1}(y)$ follows from (i). Thus z also converges to x. In other words z and x represent the same point in βX. Hence $(z, y)=(x, f(x))\in X'$. Thus (1) is confirmed.

Now we are going to prove that X' is G_δ in $W\times M$. For each $y\in M$ and each natural number n we define an open set $M_n(y)$ of $\beta X\times M$ by
$$M_n(y)=(f^{-1}(S_n(y)))^{\sim}\times S_n(y),$$
where $S_n(y)$ denotes the spherical nbd of y with radius $1/n$. Also put
$$M_n=[\cup\{M_n(y)\mid y\in M\}]\cap(W\times M).$$
Then M_n is an open set of $W\times M$ satisfying $M_n\supset X'$. Now we claim that $X'=\bigcap_{n=1}^{\infty}M_n$. To prove our claim, let $(z', y')\in W\times M-X'$. Then z' is a maximal zero set Cauchy filter satisfying $f^{-1}(y')\notin z'$ (see [1]). Let $\mathcal{F}$ be a maximal closed Cauchy filter containing z' as a subcollection. Then, since $f^{-1}(y')\notin\mathcal{F}$ is obvious, by (ii) there is $G\in\mathcal{F}$ such that $y'\notin\overline{f(G)}$ in M. Since $f^{-1}(\overline{f(G)})$ is a zero set belonging to $\mathcal{F}$, it belongs to z', and therefore we may assume without loss of generality that $G\in z'$. There is n for which $S_n(y')\cap f(G)=\emptyset$. Then we can prove that $(z', y')\notin M_{3n}(y)$ for every $y\in M$, and accordingly $(z', y')\notin M_{3n}$. Because, if $\rho(y', y)\geqq 1/3n$, then $y'\notin S_{3n}(y)$, and hence $(z', y')\notin M_{3n}(y)$. On the other hand, if $\rho(y', y)<1/3n$, then $S_{3n}(y)\cap f(G)=\emptyset$ in M, which implies that $f^{-1}(S_{3n}(y))\cap G=\emptyset$ in X. Hence $X-f^{-1}(S_{3n}(y))\in z'$ follows, i.e. $z'\notin(f^{-1}(S_{3n}(y)))^{\sim}$. Therefore $(z', y')\notin M_{3n}(y)$. Thus in any case $(z', y')\notin M_{3n}(y)$ is proved. After all we have proved $X'=\bigcap_{n=1}^{\infty}M_n$. Thus X' is G_δ in $W\times M$ which is G_δ in $\beta X\times M$. This completes our proof.

Theorem 2. *A Tychonoff space X is a G_δ-space if and only if there are sequences $\{\mathcal{W}_i \mid i=1, 2, \cdots\}$ and $\{\mathcal{U}_i \mid i=1, 2, \cdots\}$ of open covers of X such that*

⟨i⟩ $\mathcal{U}_{i+1}^{*}<\mathcal{U}_i$, $i=1, 2, \cdots$,

⟨ii⟩ *if $\mathcal{F}$ is a maximal closed filter and if for a fixed point x of X and for each i there is $F_i\in\mathcal{F}$ and $W_i\in\mathcal{W}_i$ satisfying $F_i\subset W_i\cap S(x, \mathcal{U}_i)$, then $\mathcal{F}$ converges.*

Obviously ⟨ii⟩ *is equivalent with*

⟨ii⟩′ *if $\mathcal{F}$ is a closed collection with finite intersection property and if for a fixed point x of X and for each i there is $F_i\in\mathcal{F}$ and $W_i\in\mathcal{W}_i$ satisfying $F_i\subset W_i\cap S(x, \mathcal{U}_i)$, then $\cap\{F\mid F\in\mathcal{F}\}\neq\emptyset$.*

Proof. Assume that X is a G_δ-space; then it satisfies (i) and (ii) of Theorem 1. Let $\mathcal{V}_1, \mathcal{V}_2, \cdots$ be a normal sequence of open covers of M with mesh $\mathcal{V}_i\rightarrow 0$. Put $U_n=\{f^{-1}(V)\mid V\in\mathcal{V}_n\}$. Then ⟨i⟩ of the present Theorem is satisfied. Suppose $\mathcal{F}$ is a maximal closed filter in X satisfying the condition in $\langle \mathrm{ii} \rangle$. Let $f(x)=y$. Note that $\mathcal{F}$ is Cauchy w.r.t. $\{W_n\}$. Hence if $f^{-1}(y)\in\mathcal{F}$, then $\mathcal{F}$ converges because of (i). If $f^{-1}(y)\notin\mathcal{F}$, then by (ii) $y\notin\overline{f(G)}$ for some $G\in\mathcal{F}$. Hence $S(y,\mathcal{V}_n)\cap f(G)=\emptyset$ for some n. This implies that $S(x, U_n)\cap G=\emptyset$ in X. This, however, contradicts that $F_n\subset S(x,\mathcal{U}_n)$ for $F_n\in\mathcal{F}$. Thus $\langle \mathrm{ii} \rangle$ is proved.

Conversely, assume that X satisfies $\langle \mathrm{i} \rangle$ and $\langle \mathrm{ii} \rangle$. Then X is decomposed into the disjoint union of sets of the form $\bigcap_{n=1}^{\infty} S(x,\mathcal{U}_n)$. Let us denote by M the decomposition (=disjoint closed covering) and by f the natural map from X onto M. Now, for each $y\in M$ we define that $\{f(S(f^{-1}(y),\mathcal{U}_n))\mid n=1,2,\cdots\}$ is a nbd base of y in M. Then M turns out to be a metrizable space, and f is a continuous map from X onto M. To prove (i) and (ii), we assume that $\mathcal{F}$ is a maximal closed filter in X, Cauchy w.r.t. $\{\mathcal{W}_n\}$. If $f^{-1}(y)\in\mathcal{F}$, then since $f^{-1}(y)\subset S(x,\mathcal{U}_n)$, $n=1,2,\cdots$ for any $x\in f^{-1}(y)$, by $\langle \mathrm{ii} \rangle$ $\mathcal{F}$ converges. Namely $f^{-1}(y)$ is complete. Next, assume that $f^{-1}(y)\notin\mathcal{F}$ and that $F_n\subset W_n\in\mathcal{W}_n$, $F_n\in\mathcal{F}$, $n=1,2,\cdots$. Fix a point x of $f^{-1}(y)$; then we claim that $S(x,\mathcal{U}_n)\cap G=\emptyset$ for some n and for some $G\in\mathcal{F}$. Because, if we assume the contrary, then $\overline{S(x,\mathcal{U}_{n+1})}\in\mathcal{F}$, $n=1,2,\cdots$. Therefore $F_n\cap\overline{S(x,\mathcal{U}_{n+1})}$ is a member of $\mathcal{F}$ contained in $W_n\cap S(x,\mathcal{U}_n)$. Thus by $\langle \mathrm{ii} \rangle$ $\mathcal{F}\to p$ for some point p of X. Since $f^{-1}(y)\notin\mathcal{F}$, $p\notin f^{-1}(y)$. But this implies that $S(x,\mathcal{U}_n)\cap S(p,\mathcal{U}_n)=\emptyset$ for some n and eventually $S(x,\mathcal{U}_n)\cap G=\emptyset$ for some $G\in\mathcal{F}$ contradicting our assumption. Thus $S(x,\mathcal{U}_n)\cap G=\emptyset$ for some $G\in\mathcal{F}$. This implies that $S(f^{-1}(y),\mathcal{U}_{n+1})\cap S(G,\mathcal{U}_{n+1})=\emptyset$, and hence $f(S(f^{-1}(y),\mathcal{U}_{n+1})\cap f(G)=\emptyset$, i.e. $y\notin\overline{f(G)}$, which proves that f is closed w.r.t. $\{\mathcal{W}_n\}$.

Theorem 3. *A space X is a G_δ-space if and only if it is homeomorphic to a closed set in the product of a metric space and a Čech topologically complete space.*

Proof. Theorem 2 indicates that every closed set of a G_δ-space is a G_δ-space, and thus the if part of the present Theorem follows. On the other hand it is easy to see that X' in the proof of Theorem 1 is a closed set in $W\times M$, and thus the only if part follows. Details will be left to the reader.

References

[1] J. Nagata: Modern General Topology. Amsterdam-Groningen (1968).
[2] ——: On G_δ-sets in the product of a metric space and a compact space. I. Proc. Japan Acad., **49**, 179–182 (1973).