56. An Inequality for 4-Dimensional Kählerian Manifolds

By Shûkichi TANNO
Mathematical Institute, Tôhoku University, Sendai

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1. Introduction. Let (M, g, J) be a Kählerian manifold with almost complex structure J and Kählerian metric tensor g. By $R=(R^i_{jkl})$, $(R_{jk})=(R^r_{jkr})$, and S we denote the Riemannian curvature tensor, the Ricci curvature tensor, and the scalar curvature, respectively. By dM we denote the volume element of (M, g, J). By $\chi(M)$ we denote the Euler-Poincaré characteristic of M. By $\mathrm{Vol}(M)$ we denote the total volume of (M, g, J).

Main theorem. *Let (M, g, J) be a (real) 4-dimensional compact Kählerian manifold. Then the following inequality holds:*

$$\chi(M) \geq \frac{1}{96\pi^2}\left[\int S^2 dM - 6(2-\beta)\int [R_{ij}-(S/4)g_{ij}][R^{ij}-(S/4)g^{ij}]dM\right], \tag{1.1}$$

where β is an arbitrary constant <1. The equality holds if and only if (M, g, J) is of constant holomorphic sectional curvature.

Furthermore, if (M, g, J) is an Einstein space, then

$$96\pi^2\chi(M) \geq S^2\,\mathrm{Vol}(M) \tag{1.2}$$

holds. The equality holds, if and only if (M, g, J) is of constant holomorphic sectional curvature.

We give an outline of the proof. First we need to find out inequalities concerning $(R_{ijkl}R^{ijkl})$, $(R_{jk}R^{jk})$ and S^2, such that the equality implies constancy of holomorphic sectional curvature. For this purpose we give a new characterization of the Weyl's conformal curvature tensor in § 3, and in the next section we give a characterization of the Bochner curvature tensor. In this process we have the best inequality (4.14).

2. Preliminaries. Let (M, g) be a Riemannian manifold of dimension m. By ∇ we denote the Riemannian connection with respect to g. If $R_{ijkl}=k(g_{jk}g_{il}-g_{jl}g_{ik})$ holds on M (at x, resp.) for a real number k, (M, g) is said to be of constant curvature (at x, resp.). We put

$$A(g)=R_{ijkl}R^{ijkl}-(2/(m-1))R_{jk}R^{jk}, \tag{2.1}$$

$$B(g)=R_{jk}R^{jk}-(1/m)S^2. \tag{2.2}$$

Then $A(g)\geq 0$ holds; the equality holds on M (at x, resp.) if and only if (M, g) is of constant curvature (at x, resp.). $B(g)\geq 0$ holds; the equality on M is equivalent to the fact that (M, g) is an Einstein space (cf.

for example, Barger [3]).

A (1,3)-tensor field $D=(D^i_{jkl})$ is called curvature-like, if

[i] $D^i_{jkl}=-D^i_{jlk}$,

[ii] $D_{ijkl}=D_{klij}$ (where $D_{ijkl}=g_{ih}D^h_{jkl}$),

[iii] $D_{ijkl}+D_{iklj}+D_{iljk}=0$,

[iv] $\nabla_h D_{ijkl}+\nabla_k D_{ijlh}+\nabla_l D_{ijhk}=0$.

The Riemannian curvature tensor R satisfies [i]$\sim$[iv]. If a tensor field D satisfies [i], [ii] and [iii], then we call D a *semi-curvature-like tensor field.* For brevity we treat D^i_{jkl} in the covariant form $D_{ijkl}=g_{ih}D^h_{jkl}$. If a tensor field D is expressed as a sum of tensor fields each of which contains just one of R_{****} (the Riemannian curvature tensor), R_{**} (the Ricci curvature tensor) and S, then we say that D is of *curvature degree* 1.

Proposition 2.1. *In a Riemannian manifold (M,g), every semi-curvature-like tensor field D of curvature degree* 1 *which is constructed by $(R_{****}, R_{**}, S, g_{**})$ is of the form:*

$$D_{ijkl}=aR_{ijkl}+b(R_{jk}g_{il}-R_{jl}g_{ik}+g_{jk}R_{il}-g_{jl}R_{ik}) + c(g_{jk}g_{il}-g_{jl}g_{ik})S, \tag{2.3}$$

where a, b, c are scalars on M.

3. A characterization of the Weyl's conformal curvature tensor. The Weyl's conformal curvature tensor $C=(C^i_{jkl})$, $C_{ijkl}=g_{ih}C^h_{jkl}$, is given by

$$C_{ijkl}=R_{ijkl}+b(R_{jk}g_{il}-R_{jl}g_{ik}+g_{jk}R_{il}-g_{jl}R_{ik}) + c(g_{jk}g_{il}-g_{jl}g_{ik})S, \tag{3.1}$$

where $b=-1/(m-2)$ and $c=1/(m-1)(m-2)$.

Proposition 3.1. *Let D be a tensor field defined by* (2.3). *Then the following conditions* (P) *and* (Q) *are equivalent.*

(P) *$D=0$ at x, if and only if (M,g) is of constant curvature at x,*

(Q) *$a+2(m-1)b+m(m-1)c=0$, $a\neq 0$, $a+(m-2)b\neq 0$ at x,*

We notice that the Weyl's conformal curvature tensor satisfies $a+(m-2)b=0$. If D is a tensor field defined by (2.3) and satisfies (P) or equivalently (Q), then the inner product $(D,D)=(D_{ijkl}D^{ijkl})$ is given by

$$\begin{aligned}(D,D)&=a^2R_{ijkl}R^{ijkl}+[8ab+4(m-2)b^2]R_{jk}R^{jk}\\ &\quad+[4ac+4b^2+8(m-1)bc+2m(m-1)c^2]S^2\\ &=a^2A(g)+[2a^2/(m-1)+8ab+4(m-2)b^2]B(g).\end{aligned} \tag{3.2}$$

For a Riemannian manifold (M,g), we define $\mathcal{D}$ and $\mathcal{D}_0$ by

$\mathcal{D}=$[the set of all semi-curvature-like tensor fields of curvature degree 1 which are constructed by $(R_{****}, R_{**}, S, g_{**})$ such that $a=1$].

$\mathcal{D}_0=$[the subset of $\mathcal{D}$ composed of elements D such that $D=0$ is equivalent to the fact that (M,g) is of constant curvature].

Then $D\in\mathcal{D}_0$ is denoted by the parameter b. For any element D

$=D(b)\in\mathcal{D}_0$, we have

(3.3) $$(D,D)=A(g)+[2/(m-1)+8b+4(m-2)b^2]B(g)\geq 0.$$

The coefficient of $B(g)$ satisfies

(3.4) $$2/(m-1)+8b+4(m-2)b^2>-2m/(m-1)(m-2).$$

In (3.4), (the left hand side)-(the right hand side)$\to 0$ as $b\to -1/(m-2)$.

Theorem 3.2. *In a Riemannian manifold (M,g), the Weyl's conformal curvature tensor C is characterized by $C\in\mathcal{D}$ such that*

$C=$*the* $\lim_{(b)}$ *of* $\{D(b)\in\mathcal{D}_0\}$ *such that* $(D(b),D(b))\to\inf$.

4. A characterization of the Bochner curvature tensor. Let (M,g,J) be a Kählerian manifold. J and g satisfy

(4.1) $$g_{rs}J_i^rJ_j^s=g_{ij},\qquad J_r^iJ_j^r=-\delta_j^i$$

and $\nabla_hJ_j^i=0$. We need the following identities (cf. Yano [10]):

(4.2) $$R_{ijkl}J_r^kJ_s^l=R_{ijrs},\qquad R_{ijks}J_r^k=-R_{ijrk}J_s^k,$$

(4.3) $$R_{ij}J_r^iJ_s^j=R_{rs},\qquad R_{ir}J_j^r=-R_{jr}J_i^r,$$

(4.4) $$R_{ijkl}J^{kl}=2J_i^rR_{rj},$$

(4.5) $$2R_{ijkl}J^{jl}=R_{ikjl}J^{jl},$$

where $J^{jk}=J_r^jg^{rk}$ and $J_{rs}=g_{rt}J_s^t$.

As a proposition similar to Proposition 2.1, after some complicated calculations, we have

Proposition 4.1. *In a Kählerian manifold (M,g,J) every semi-curvature-like tensor field D of curvature degree 1 which is constructed by $(R_{****},R_{**},S,g_{**},J_*^*)$ is of the form:*

(4.6) $$\begin{aligned}D_{ijkl}=&aR_{ijkl}+b(R_{jk}g_{il}-R_{jl}g_{ik}+g_{jk}R_{il}-g_{jl}R_{ik})\\&+c(R_{jr}J_k^rJ_{il}-R_{jr}J_l^rJ_{ik}+J_{jk}R_{ir}J_l^r-J_{jl}R_{ir}J_k^r\\&\quad-2J_{ij}R_{kr}J_l^r-2J_{kl}R_{ir}J_j^r)\\&+d(J_{jk}J_{il}-J_{jl}J_{ik}-2J_{ij}J_{kl})S+e(g_{jk}g_{il}-g_{jl}g_{ik})S,\end{aligned}$$

where a, b, c, d, e are scalars on M.

The Bochner curvature tensor $B=(B_{jkl}^i)$ is given by (cf. Tachibana [7], Bochner [5])

(4.7) $$\begin{aligned}B_{jkl}^i=&R_{jkl}^i-(1/(m+4))(R_{ik}\delta_l^i-R_{jl}\delta_k^i+g_{jk}R_l^i-g_{jl}R_k^i\\&+R_{jr}J_k^rJ_l^i-R_{jr}J_l^rJ_k^i+J_{jk}R_r^iJ_l^r-J_{jl}R_r^iJ_k^r\\&-2R_{kr}J_l^rJ_j^i-2R_r^iJ_j^rJ_{kl})\\&+(1/(m+2)(m+4))(g_{jk}\delta_l^i-g_{jl}\delta_k^i+J_{jk}J_l^i-J_{jl}J_k^i-2J_{kl}J_j^i)S.\end{aligned}$$

A Kählerian manifold (M,g,J), $m\geq 4$, is of constant holomorphic sectional curvature H at x if and only if

(4.8) $$R_{ijkl}=(H/4)[(g_{il}g_{jk}-g_{ik}g_{jl})+(J_{il}J_{jk}-J_{ik}J_{jl}-2J_{ij}J_{kl})]$$

holds at x for a real number H. Then R_{jk} and S are given by

(4.9) $$4R_{jk}=(m+2)Hg_{jk},\qquad 4S=m(m+2)H.$$

Subtracting the right hand side from the left hand side of (4.8), applying $(4.9)_2$, and taking the inner product $E(g,J)$ with itself, we have an inequality:

(4.10) $$E(g,J)=R_{ijkl}R^{ijkl}-[8/m(m+2)]S^2\geq 0.$$

The equality holds (at x, resp.) if and only if (M, g, J) is of constant holomorphic sectional curvature (at x, resp.).

Proposition 4.2. *Let D be a tensor field defined by* (4.6). *Then the following conditions* (P*) *and* (Q*) *are equivalent*:

(P*) $D=0$ *at x, if and only if (M, g, J) is of constant holomorphic sectional curvature at x,*

(Q*) $$a+2(m-1)b+6c+3md+m(m-1)e=0,$$
$$(m+2)(2b+me)=-a=(m+2)(2c+md),$$
$$a\neq 0, \quad a+(m-2)b+6c\neq 0 \quad \textit{hold at } x.$$

Let D be a tensor field defined by (4.6) satisfying (P*) or equivalently (Q*). Then we have

(4.11) $$(D, D)=aE(g, J) +[8ab+24ac+4(m-2)b^2+48bc+12(m+2)c^2]B(g)\geq 0.$$

For a Kählerian manifold (M, g, J) we define $\mathscr{D}^*$ and $\mathscr{D}_0^*$ by

$\mathscr{D}^*=$[the set of all semi-curvature-like tensor fields of curvature degree 1 constructed by $(R_{****}, R_{**}, S, g_{**}, J_*^*)$ such that $a=1$].

$\mathscr{D}_0^*=$[the subset of $\mathscr{D}^*$ composed of elements D such that $D=0$ is equivalent to the fact that (M, g, J) is of constant holomorphic sectional curvature].

For $a=1$, the coefficient of $B(g)$ of (4.11) satisfies

(4.12) $$8b+24c+4(m-2)b^2+48bc+12(m+2)c^2>-16/(m+4).$$

In (4.12), (the left hand side)-(the right hand side)$\to 0$ as $b, c\to -1/(m+4)$ for $m\geq 6$; as $2b+6c\to -1$ for $m=4$.

Theorem 4.3. *The Bochner curvature tensor is characterized by*

(1) *for $m\geq 6$, $B=$the* $\lim_{(b,c)}$ *of* $\{D(b, c)\in \mathscr{D}_0^*\}$ *such that* $(D, D)\to \inf$,

(2) *for $m=4$, $B=$the* $\lim_{(b=c)}$ *of* $\{D(b, c)\in \mathscr{D}_0^*\}$ *such that* $(D, D)\to \inf$.

Theorem 4.4. *In a Kählerian manifold (M, g, J) we have*

(4.13) $$E(g, J)-[16\beta/(m+4)]B(g)\geq 0, \quad \textit{i.e.},$$

(4.14) $$R_{ijkl}R^{ijkl}-[16\beta/(m+4)]R_{jk}R^{jk} +[(16(m+2)\beta-8(m+4))/m(m+2)(m+4)]S^2\geq 0,$$

where β is a constant <1. The equality holds on M (at x, resp.), if and only if (M, g, J) is of constant holomorphic sectional curvature (at x, resp.).

5. Euler-Poincaré characteristics of 4-dimensional compact Kählerian manifolds. Let (M, g, J) be a (real) 4-dimensional compact Kählerian manifold. Every Kählerian manifold is orientable. Then the Gauss-Bonnet formula is

(5.1) $$\int [R_{ijkl}R^{ijkl}-4R_{jk}R^{jk}+S^2]dM=32\pi^2\chi(M)$$

(cf. for example, Berger [3]). Integrating (4.14) with $m=4$, we have

(5.2) $$\int [R_{ijkl}R^{ijkl}-2\beta R_{jk}R^{jk}+((3\beta-2)/6)S^2]dM\geq 0.$$

Eliminating $R_{ijkl}R^{ijkl}$ from (5.1) and (5.2) we have the main theorem.

6. Remarks. (I) The Riemannian case of (1.2) is (cf. Avez [2], Bishop-Goldberg [4]): For a compact orientable Einstein space (M, g), $m=4$,

$$192\pi^2\chi(M)\geq S^2 \operatorname{Vol}(M),$$

where the equality holds if and only if (M, g) is of constant curvature.

(II) For the Riemannian case of (1.1), see Tanno [9].

(III) For the Bochner curvature tensor, see [5], [7], [8], etc.

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