50. *Cauchy Problem for Degenerate Parabolic Equations*

By Katsuju IGARI
Department of Mathematics, Kyoto University

(Comm. by Kôsaku YOSIDA, M. J. A., April 12, 1973)

1. Introduction. We consider the Cauchy problem for the equation

$$\partial_t u - \sum_{j,k=1}^{n} \partial_{x_j}(a_{jk}(x,t)\partial_{x_k}u) - \sum_{j=1}^{n} b_j(x,t)\partial_{x_j}u - c(x,t)u = \partial_t u - Au = f, \tag{1.1}$$

(x,t) in $\boldsymbol{R}^n \times [0,\infty)$ with the initial-value

$$u(x,0) = u_0(x), \tag{1.2}$$

where $a_{jk}(x,t)$, $b_j(x,t)$, $c(x,t)$ are real-valued smooth functions. We assume that $(a_{jk})_{1\le j\le n, 1\le k\le n}$ is symmetric and satisfies the condition: for any $(x,t)\in \boldsymbol{R}^n\times[0,\infty)$

$$\sum_{j,k=1}^{n} a_{jk}(x,t)\xi_j\xi_k \ge 0 \qquad \text{for all } \xi \in \boldsymbol{R}^n. \tag{1.3}$$

O. A. Oleĭnik has treated this problem (see [3] and [4]). Her method consists of the following procedure (elliptic regularization): Instead of (1.1), the following equations (depending on a positive parameter ε) in $G=\boldsymbol{R}^n\times[0,T]$

$$\partial_t u - \varepsilon\Delta u - Au = f \tag{1.4}$$

are considered. Let u_ε be the solution of (1.4) with the given initial-value $u_0(x)\in L^2(\boldsymbol{R}^n)$ and $f(x,t)\in L^2(G)$. Then it is shown that $\{u_\varepsilon(x,t)\}$ is bounded in $L^2(G)$. Then a weak limit of them, as $\varepsilon\to+0$, gives the desired solution $u(x,t)\in L^2(G)$. The uniqueness of the solution is proved. She also proved the smoothness of u, assuming the smoothness of u_0 and f.

Contrary to the above point of view, we regard (1.1) as evolution equation. More precisely, we want to show the existence of the unique solution $u(x,t)\in \mathcal{E}_t^0(L^2)\cap\mathcal{E}_t^1(\mathcal{D}_{L^2}'^2)$ of (1.1)–(1.2) for any $f(x,t)\in\mathcal{E}_t^0(L^2)$ and any initial-value $u_0(x)\in L^2$.*)

Our approach is based on the semi-group theory. Instead of elliptic regularization, we use Friedrichs' mollifier. Its property (see

*) Throughout this paper, we use the following notation: $x=(x_1,\cdots,x_n)$. $\partial_t=\partial/\partial t$, $\partial_j=\partial x_j=\partial/\partial x_j$, $\partial_x^\nu=\partial_1^{\nu_1}\cdots\partial_n^{\nu_n}$, where $\nu=(\nu_1,\cdots,\nu_n)$. $L^2=L^2(\boldsymbol{R}^n)$. $u(x)\in\mathcal{D}_{L^2}^m$ means that its derivatives (in the sense of distribution) $\partial_x^\nu u$ up to order m belong to L^2. $\mathcal{D}_{L^2}'^m$ is the dual space of $\mathcal{D}_{L^2}^m$ and sometimes we denote it by $\mathcal{D}_{L^2}^{-m}$. $\varphi(x)\in\mathcal{B}^m$ means that its derivatives $\partial_x^\nu\varphi$ up to order m are continuous and bounded in $\boldsymbol{R}^n$. $f(t)\in\mathcal{E}_t^k(\mathcal{D}_{L^2}^m$ (or $\mathcal{B}^m$)) means that $t\to f(t)\in\mathcal{D}_{L^2}^m$ (or $\mathcal{B}^m$) is continuously differentiable up to order k.

Lemma) gives immediately the desired result (see Theorem 1). The smoothness can be obtained in the following form: when $u_0(x) \in \mathcal{D}^m_{L^2}$ and $f(x,t) \in \mathcal{E}^0_t(\mathcal{D}^m_{L^2})$, the solution $u(x,t)$ belongs to $\in \mathcal{E}^0_t(\mathcal{D}^m_{L^2}) \cap \mathcal{E}^1_t(\mathcal{D}^{m-2}_{L^2})$.

It seems to us that our method is more natural than the one relying on elliptic regularization and will be useful to other problems. A forthcoming paper will give the detailed proof including some other results.

2. Statement of results. Let $a_{jk}(x,t) \in \mathcal{E}^0_t(\mathcal{B}^2)$; $b_j(x,t) \in \mathcal{E}^0_t(\mathcal{B}^1)$; $c(x,t) \in \mathcal{E}^0_t(\mathcal{B}^0)$. We assume the condition (1.3). Then we have the following theorem.

Theorem 1. *For any initial-value $u_0(x) \in L^2$ and any $f(x,t) \in \mathcal{E}^0_t(L^2)$, there exists a unique solution $u(x,t) \in \mathcal{E}^0_t(L^2) \cap \mathcal{E}^1_t(\mathcal{D}'^2_{L^2})$ of the Cauchy problem* (1.1)–(1.2).

To prove this, following propositions are essential. The first one is the energy inequality. The second one shows that Hille-Yosida's theorem is applicable.

Proposition 1. *Let $f(x,t) \in \mathcal{E}^0_t(L^2)$ and $u(x,t) \in \mathcal{E}^0_t(L^2) \cap \mathcal{E}^1_t(\mathcal{D}'^2_{L^2})$ be the solution of* (1.1). *Then it holds for any t $(0 \le t \le T)$*

$$\|u(t)\| \le e^{\gamma t}\|u(0)\| + \int_0^t e^{\gamma(t-s)}\|f(s)\|\,ds, \tag{2.1}$$

where γ is a constant which may depend on T but does not depend on u and f.

Now we assume coefficients be functions of only x. Then we can obtain the following proposition.

Proposition 2. *Take the domain of definition $\mathcal{D}(A)$ of A as follows:*

$$\mathcal{D}(A) = \{u\,;\, u \in L^2, Au \in L^2\}. \tag{2.2}$$

Then, for large λ, $(\lambda - A)$ defines a one-to-one surjective mapping of $\mathcal{D}(A)$ onto L^2. Moreover there exists a constant β such that

$$\|(\lambda - A)^{-1}\|_{\mathcal{L}(L^2,L^2)} \le \frac{1}{\lambda - \beta} \qquad \text{for any } \lambda > \beta. \tag{2.3}$$

If we use the following lemma, these propositions can be proved in the same way as hyperbolic equation (see [2], §§2, 4 in Chapter 6).

Lemma. *Let $\rho_\varepsilon *$ be Friedrichs' mollifier, where we assume $\rho(x)$ even function. Let $a(x) \in \mathcal{B}^2$ be real-valued function, and let $u(x) \in L^2$. Then it holds for any $\nu (|\nu| \le 2)$*

1) $$|\mathrm{Re}\,(u_\varepsilon, [\rho_\varepsilon *, a(x)]\partial_x^\nu u)| \le C\|u\|,$$

2) $$\mathrm{Re}\,(u_\varepsilon, [\rho_\varepsilon *, a(x)]\partial_x^\nu u) \to 0 \qquad \text{as } \varepsilon \to +0,$$

where u_ε stands for $\rho_\varepsilon * u$, $[\rho_\varepsilon *, a(x)]\partial_x^\nu u = \rho_\varepsilon * \{a(x)\partial_x^\nu u\} - a(x)\rho_\varepsilon * \partial_x^\nu u$, and C is a constant independent of u and ε.

Proof of Lemma. Consider only the case of $|\nu| = 2$, because, in the case of $|\nu| \le 1$, 1) and 2) are clear by Friedrichs' lemma. We denote ∂_x^ν by $\partial_j \partial_k$. By Taylor expansion

$$[\rho_\varepsilon *, a(x)]\partial_j\partial_k u = -\sum_{i=1}^{n} a^{(i)}(x)(x_i\rho_\varepsilon)*\partial_j\partial_k u \tag{2.4}$$
$$+\sum_{|\mu|=2}\frac{1}{\mu!}\int a_\mu(x,y)(x-y)^\mu \rho_\varepsilon(x-y)\partial_{y_j}\partial_{y_k}u(y)dy.$$

At first take the 1-st term of (2.4).

$$-2\,\mathrm{Re}\,(u_\varepsilon, a^{(i)}(x)(x_i\rho_\varepsilon)*\partial_j\partial_k u) = (a^{(i,j)}(x)(x_i\rho_\varepsilon)*\partial_k u, u_\varepsilon) \tag{2.5}$$
$$+(u_\varepsilon, [(x_i\rho_\varepsilon)*, a^{(i)}(x)]\partial_j\partial_k u) + (u_\varepsilon, (x_i\rho_\varepsilon)*\{a^{(i,k)}(x)\partial_j u\})$$
$$+((x_i\rho_\varepsilon)*\partial_k u, [a^{(i)}(x), \rho_\varepsilon *]\partial_j u),$$

where we used the relations that $((x_i\rho_\varepsilon)*u, v) = -(u, (x_i\rho_\varepsilon)*v)$ and that $(\rho_\varepsilon * u, v) = (u, \rho_\varepsilon * v)$. These all terms in the right-hand side of (2.5) can be majorized by

$$\sum_{i,j=1}^{n}\|u\|\,\|(x_i\rho_\varepsilon)*\partial_j u\| + \sum_{|\nu|\le 2}\sum_{i=1}^{n}\|u\|\,\|[a^{(i)}(x), (x_i\rho_\varepsilon)*]\partial_x^\nu u\|.$$

In the same way as Friedrichs' lemma, we can show for $\nu(|\nu|\le 2)$

$$\|(x_i\rho_\varepsilon)*\partial_j u\|,\ \|[a^{(i)}(x), (x_i\rho_\varepsilon)*]\partial_x^\nu u\| \le \text{const.}\ \|u\|,$$
$$\|(x_i\rho_\varepsilon)*\partial_j u\|,\ \|[a^{(i)}(x), (x_i\rho_\varepsilon)*]\partial_x^\nu u\| \to 0 \qquad \text{as } \varepsilon \to +0.$$

Next we consider the 2-nd term of (2.4). Denote it by $R_\varepsilon u$.

$$R_\varepsilon u = \sum_{|\mu|=2}\frac{1}{\mu!}\int \partial_{y_k}\partial_{y_j}\{a_\mu(x,y)(x-y)^\mu\rho_\varepsilon(x-y)\}u(y)dy \tag{2.6}$$
$$= \sum_{|\mu|=2}\frac{1}{\mu!}\int \partial_{y_k}\partial_{y_j}\{a_\mu(x,y)(x-y)^\mu\rho_\varepsilon(x-y)\}\{u(y)-u(x)\}dy.$$

If we note that $\sum_{|\nu|\le 2}\sum_{|\mu|=2}\int |(x^\mu\rho_\varepsilon)^{(\nu)}(x)|\,dx <$ const. (independent of ε), the same reasoning as in the proof of Friedrichs' lemma gives

$$\|R_\varepsilon u\| \le \text{const.}\ \|u\|,$$
$$R_\varepsilon u \to 0 \qquad \text{as } \varepsilon \to 0.$$

Thus the proof is completed.

At the end we state the theorem concerning the smoothness of the solution. Let $a_{jk}(x,t) \in \mathcal{E}_t^0(\mathcal{B}^{m+2})$; $b_j(x,t) \in \mathcal{E}_t^0(\mathcal{B}^{m+1})$; $c(x,t) \in \mathcal{E}_t^0(\mathcal{B}^m)$, where $m = 0, 1, 2, \cdots$. We assume the condition (1.3). Then we have the following theorem.

Theorem 2. *For any initial-value* $u_0(x) \in \mathcal{D}_{L^2}^m$ *and any* $f(x,t) \in \mathcal{E}_t^0(\mathcal{D}_{L^2}^m)$, *there exists a unique solution* $u(x,t) \in \mathcal{E}_t^0(\mathcal{D}_{L^2}^m) \cap \mathcal{E}_t^1(\mathcal{D}_{L^2}^{m-2})$ *of the Cauchy problem* (1.1)–(1.2).

Acknowledgement. The author should like to express his sincere thanks to Professor S. Mizohata for many valuable comments and for the constant encouragement.

References

[1] K. O. Friedrichs: Symmetric hyperbolic system of linear differential equations. Comm. P. A. M., **7**, 345–392 (1954).

[2] S. Mizohata: Theory of Partial Differential Equations. Iwanami, Tokyo (1965) (in Japanese; will appear in English from Camb. Univ. press).

[3] O. A. Oleĭnik: On the smoothness of the solutions of degenerate elliptic and parabolic equations. Sov. M. Dokl., **6**, 972–976 (1965).

[4] ——: Linear equations of second order with non-negative form. Mat. Sbornik, **69**, 111–140 (1966) (in Russian). Amer. M. Soc. Translation, 167–199.