89. *On Normal Approximate Spectrum. IV*

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1. Introduction. In our previous notes [3], [5], [6] and [7], we have discussed some properties of the normal approximate spectra of operators on Hilbert space $\mathfrak{H}$. A complex number λ is an *approximate propervalue* of an operator T on $\mathfrak{H}$ if there is a sequence of unit vectors in $\mathfrak{H}$ such that

(*) $$\|(T-\lambda)x_n\|\to 0 \qquad (n\to\infty).$$

Then sequence $\{x_n\}$ is called *approximate propervectors* belonging to λ. The set $\pi(T)$ of all approximate propervalues is called the *approximate spectrum* of T. If there is a sequence $\{x_n\}$ of unit vectors for λ and T satisfying (*) and

(**) $$\|(T-\lambda)^*x_n\|\to 0 \qquad (n\to\infty),$$

the λ is called a *normal approximate propervalue* of T and $\{x_n\}$ *normal approximate propervectors*. The set $\pi_n(T)$ of all normal approximate propervalues of T is called the *normal approximate spectrum* of T. Some equivalent conditions are discussed in [3], [5] and [7].

In the present note, we shall prove three theorems in terms of the normal approximate spectra in §§ 3–5. In the proofs, we shall use the Berberian representation in [1], which is sketched in § 2.

2. The Berberian representation. Let $\mathfrak{B}$ be the set of all bounded sequences of vectors of $\mathfrak{H}$. Then $\mathfrak{B}$ is a vector space with respect to the operations:

$$\{x_n\}+\{y_n\}=\{x_n+y_n\}$$

and

$$\alpha\{x_n\}=\{\alpha x_n\}.$$

Let (for a fixed Banach limit Lim)

$$\mathfrak{N}=\{\{x_n\}\in\mathfrak{B}\,;\ \operatorname*{Lim}_{n\to\infty}(x_n|y_n)=0 \text{ for all } y_n\in\mathfrak{B}\},$$

and let $\mathfrak{P}=\mathfrak{B}/\mathfrak{N}$. Then $\mathfrak{P}$ becomes an inner product space by

$$(\{x_n\}+\mathfrak{N}|\{y_n\}+\mathfrak{N})=\operatorname*{Lim}_{n\to\infty}(x_n|y_n).$$

If $x\in\mathfrak{H}$, then $\{x\}$ means the sequence of all whose terms are x.

$$(x'|y')=(x|y)$$

for $x'=\{x\}+\mathfrak{N}$ and $y'=\{y\}+\mathfrak{N}$, so that the mapping $x\to x'$ is an isometric linear map of $\mathfrak{H}$ onto a closed subspace $\mathfrak{H}'$ of $\mathfrak{P}$. Let $\mathfrak{K}$ be the

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completion of $\mathfrak{P}$. Then $\mathfrak{K}$ is an extension of $\mathfrak{H}$. For an operator T acting on $\mathfrak{H}$, put

$$T^0(\{x_n\}+\mathfrak{N})=\{Tx_n\}+\mathfrak{N}.$$

We can extend T^0 on $\mathfrak{K}$, which will be denoted by T^0 too. The mapping $T\to T^0$ of $\mathfrak{B}(\mathfrak{H})$ into $\mathfrak{B}(\mathfrak{K})$ will be called the *Berberian representation*. The following theorem is proved in [1].

Theorem A (Berberian). *The Berberian representation is *-isomorphic and isometric. If $T\in\mathfrak{B}(\mathfrak{H})$, then*

$$\pi(T)=\pi(T^0)=\sigma_p(T^0), \tag{1}$$

where $\sigma_p(T)$ is the point spectrum of T.

3. Naked point. A point λ of a compact set S in the plane is called a *naked point* of S in the sense of [4] if there are λ_n and r_n such that

(i) $\{\mu;\,|\mu-\lambda_n|<r_n\}\subset S^c$,

(ii) $\lambda_n\to\lambda \quad (n\to\infty)$,

and

(iii) $\dfrac{|\lambda_n-\lambda|}{r_n}\to 1 \ (n\to\infty)$,

where S^c is the complement of S. The notion of naked points is originally introduced by Sz.-Nagy and Foiaş [12].

An operator T is called to satisfy the *condition* (G_1) if

$$\|(T-\lambda)^{-1}\|\leqq\frac{1}{\operatorname{dist}(\lambda,\sigma(T))} \tag{2}$$

for every $\lambda\notin\sigma(T)$ where $\sigma(T)$ is the spectrum of T.

In [4], one of the authors proved the following theorem which is an extension of a theorem of Berberian [2]:

Theorem B. *If T is an operator satisfying the condition (G_1), and if λ is a naked point of the spectrum $\sigma(T)$ and a propervalue of T, then λ is a normal propervalue of T.*

Recently, extending the notion of the condition (G_1), Saito [10] introduces the following definition:

Definition C. An operator T is called to satisfy *Saito's condition* for X if $\sigma(T)\subset X$ and

$$\|(T-\lambda)^{-1}\|\leqq\frac{1}{\operatorname{dist}(\lambda,X)} \tag{3}$$

for every $\lambda\notin X$.

Clearly, Saito's condition includes the condition (G_1) if $X=\sigma(T)$ (also it gives convexoids if X is the convex hull of $\sigma(T)$). The authors express their hearty thanks to Prof. T. Saito who gives us an opportunity to read [10] before publication.

We can prove the following extension of Theorem B without substantial change:

Theorem B′. *Let T be an operator satisfying Saito's condition* *for a closed set X. If λ is a naked point of X and a propervalue of T, then λ is a normal propervalue of T.*

Saito [10] proved the following theorem which is a generalization of Theorem B′. He used the method of unitary dilations. We shall give an alternative proof based on the Berberian representation.

Theorem 1 (Saito). *Let T be an operator satisfying Saito's condition for a closed set X. If λ is a naked point of X and an approximate propervalue of T, then λ is a normal approximate propervalue.*

Proof. Since the Berberian representation is $*$-isomorphic and isometric, $\sigma(T^0)$ coincides with $\sigma(T)$, so that T^0 satisfy Saito's condition for X too. Since λ is a propervalue of T^0 by Theorem A, Theorem B′ implies that λ is a normal propervalue of T^0. Hence there is a unit vector $z \in \mathfrak{K}$ such that $(T^0-\lambda)z=0$ and $(T^0-\lambda)^*z=0$. Therefore, there is a sequence $\{z^{(m)}\}$ of unit vectors in $\mathfrak{P}$ such that $z^{(m)}\to z$, and so $(T^0-\lambda)z^{(m)}\to 0$ and $(T^0-\lambda)^*z^{(m)}\to 0$. Let $z^{(m)}=\{x_n^{(m)}\}+\mathfrak{N}$ with $\|x_n^{(m)}\|=1$. We have

$$\varliminf_{n\to\infty}\|(T-\lambda)x_n^{(m)}\|\leqq \operatorname*{Lim}_{n\to\infty}\|(T-\lambda)x_n^{(m)}\|=\|(T^0-\lambda)z^{(m)}\|$$

and

$$\varliminf_{n\to\infty}\|(T-\lambda)^*x_n^{(m)}\|\leqq \operatorname*{Lim}_{n\to\infty}\|(T-\lambda)^*x_n^{(m)}\|=\|(T^0-\lambda)^*z^{(m)}\|$$

for $m=1,2,\cdots$. It follows that there exists a subsequence $\{x_{n(k)}^{m(k)}\}$ satisfying $(*)$ and $(**)$, so that $\lambda\in\pi_n(T)$.

4. Šilov boundary. Let A be a function algebra on a compact set X in the plane, that is, A is a Banach algebra of continuous functions on X equipped with the sup-norm which separates points of X and contains the constant functions. If there exists $f\in A$ such that $|f(\mu)|<|f(\lambda)|$ for every $\mu\neq\lambda(\mu\in X)$, then λ is called a *peak point* for A. The set of all peak points is called the *minimal boundary* for A. The closure $\partial_A X$ of the minimal boundary is the *Šilov boundary* for A.

Let X be a compact set in the plane. Let $R(X)$ be the normed algebra of all rational functions with no poles on X equipped with the sup-norm. X is called a *spectral set* for an operator T if $\sigma(T)\subset X$ and $\|f(T)\|\leqq\|f\|$ for any $f\in R(X)$, that is, the mapping $f\to f(T)$ is a contractive operator representation of $R(X)$ (cf. [9]). In the below, the uniform closure of $R(X)$ is denoted by A.

In [8], the following theorem is proved:

Theorem D (Lebow). *If S is a spectral set for an operator T, and if $\lambda\in\sigma_p(T)$ belongs to the minimal boundary for A, then λ is a normal propervalue.*

We shall extend Theorem D into the following theorem:

Theorem 2. *If S is a spectral set for T and $\lambda\in\pi(T)\cap\partial_A S$, then $\lambda\in\pi_n(T)$.*

Proof. Suppose that λ belongs to the minimal boundary for A. Then λ is a propervalue of T^0 by Theorem A, and by Theorem D λ is a normal propervalue of T^0 because S is a spectral set for T^0 too. Hence $\lambda \in \pi_n(T)$ as in the proof of Theorem 1.

Since $\pi_n(T)$ is closed as in [7] and the minimal boundary is dense in $\partial_A S$, we have $\pi(T) \cap \partial_A S \subset \pi_n(T)$.

5. Convex spectral set. The following theorem is a corollary of a theorem of Sz.-Nagy and Foiaş [12].

Theorem E. *If S is a convex spectral set for an operator T and if $\lambda \in \sigma_p(T)$ is in the (natural) boundary ∂S of S, then λ is a normal propervalue.*

Proof. Since S is convex, every $\lambda \in \partial S$ is a naked point, so that λ is a normal propervalue by a theorem of Sz.-Nagy and Foiaş [12].

We shall extend Theorem E by the use of the Berberian representation as follows:

Theorem 3. *If S is a convex spectral set for an operator T and $\lambda \in \pi(T) \cap \partial S$, then $\lambda \in \pi_n(T)$.*

Proof. Since the Berberian representation is $*$-isomorphic and isometric, S is a spectral set for T^0, and λ is a propervalue by Theorem A, so that λ is a normal propervalue of T^0 by Theorem E. Hence $\lambda \in \pi_n(T)$ as in the proof of Theorem 1.

6. An appendix. It is well-known

$$\sigma(ST)\setminus\{0\}=\sigma(TS)\setminus\{0\} \tag{4}$$

for every operators S and T. However, we can not replace σ by π_n in (4):

Proposition. *There are operators S and T such that S is* invertible and

$$\pi_n(T)\setminus\{0\}\neq\pi_n(S^{-1}TS)\setminus\{0\}. \tag{5}$$

Proof. Put

$$T=\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix} \quad \text{and} \quad R=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}.$$

Then T and R are similar by

$$S=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$$

which is clearly invertible and $TS=SR$. We have

$$\sigma(T)=\{0,1\}, \quad \sigma(R)=\pi_n(R)=\{0,1\},$$

$$T\begin{pmatrix}0\\ 1\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \quad \text{and} \quad T\begin{pmatrix}1\\ 1\end{pmatrix}=\begin{pmatrix}1\\ 1\end{pmatrix}.$$

On the other hand, we have

$$T^*\begin{pmatrix}0\\ 1\end{pmatrix}=\begin{pmatrix}1\\ 0\end{pmatrix} \quad \text{and} \quad T^*\begin{pmatrix}1\\ 1\end{pmatrix}=\begin{pmatrix}2\\ 0\end{pmatrix},$$

so that $\pi_n(T)$ is empty. Hence (5) is proved.

The above example, due to Prof. H. Choda who improves our original construction, has an another application. Let $\boldsymbol{S}$ be the set of all operators with non-void normal approximate spectra which is the uniform closure of the set $\boldsymbol{R}_1$ of all operators having one-dimensional reducing subspaces as proved by Stampfli [11]. Then $R \in \boldsymbol{R}_1 \subset \boldsymbol{S}$ and $T = SRS^{-1} \notin \boldsymbol{S}$, so that we have

Corollary. *$\boldsymbol{S}$ is not invariant under similarity.*

Remark. Prof. H. Choda also pointed out, Proposition follows from a theorem of Wogen [13] which states that a factor of type I is generated by an operator which is similar to an hermitean operator. In the above example, T generates the algebra of all 2×2 matrices as C^*-algebra which is a factor of type I_2. Hence $\pi_n(T)$ is empty by the reciprocity of the characters and the normal approximate spectrum proved in [3] and [7].

Added in proof. Sa Ge Lee (Notice A.M.S., **19**, pp. A-185-186 (1972)) announced that he obtained independently a series of similar results to our I-V.

References

[1] S. K. Berberian: Approximate proper vectors. Proc. Amer. Math. Soc., **13**, 111–114 (1962).

[2] ——: An extension of Weyl's theorem to a class of not necessarily normal operators. Mich. Math. J., **16**, 273–279 (1969).

[3] M. Enomoto, M. Fujii, and K. Tamaki: On normal approximate spectrum. Proc. Japan Acad., **48**, 211–215 (1972).

[4] M. Fujii: On some examples of non-normal operators. Proc. Japan Acad., **47**, 458–463 (1971).

[5] M. Fujii and R. Nakamoto: On normal approximate spectrum. II. Proc. Japan Acad., **48**, 297–301 (1972).

[6] M. Fujii and K. Tamaki: On normal approximate spectrum. III. Proc. Japan Acad., **48**, 389–393 (1972).

[7] I. Kasahara and H. Takai: Approximate propervalues and characters of C^*-algebras. Proc. Japan Acad., **48**, 91–93 (1972).

[8] A. Lebow: On von Neumann's theory of spectral sets. J. Math. Anal., Appl., **7**, 64–90 (1963).

[9] J. von Neumann: Eine Spektraltheorie für allgemeine Operatoren eines unitären Raumes. Math. Nachr., **4**, 258–281 (1951).

[10] T. Saito: A theorem on boundary spectra. Acta Sci. Math., Szeged, **33**, 101–104 (1972).

[11] J. G. Stampfli: On hyponormal and Toeplitz operators. Math. Ann., **183**, 328–336 (1969).

[12] B. Sz.-Nagy and C. Foiaş: Une relation parmi les vecteurs propres d'un opérateur de l'espace de Hilbert et de l'opérateur adjoint. Acta Sci. Math., Szeged, **20**, 91–96 (1959).

[13] W. R. Wogen: Von Neumann algebras generated by operators similar to normal operators. Pacif. J. Math., **37**, 539–543 (1971).