87. *Cotangential Decomposition of the Sheaf* $\mathcal{D}'/\mathcal{E}$

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The aim of this note is to construct a sheaf in the distribution theory which has analogous properties to those of the sheaf $\mathcal{C}$ important in the hyperfunction theory.

Let Ω be a domain in $\boldsymbol{R}^n$ and let $\mathcal{D}'$, $\mathcal{E}$, $\mathcal{B}$ and $\mathcal{A}$ denote the sheaves of the germs of distributions, infinitely differentiable functions, hyperfunctions and real analytic functions in Ω respectively. The quotient sheaves $\mathcal{D}'/\mathcal{E}$, $\mathcal{B}/\mathcal{A}$ and $\mathcal{D}'/\mathcal{A}$ should be called the sheaves of singularities over Ω. In 1969 M. Sato decomposed the sheaf $\mathcal{B}/\mathcal{A}$ into the cotangential directions. That is, he constructed a sheaf $\mathcal{C}$ over the cosphere bundle $S^*\Omega$ whose direct image $\pi_*\mathcal{C}$ along the projection π onto the base space Ω is isomorphic to the sheaf $\mathcal{B}/\mathcal{A}$. Actually this induces an isomorphism of global sections:

$$\mathcal{B}(\Omega)/\mathcal{A}(\Omega)\cong\pi_*\mathcal{C}(\Omega)\cong\mathcal{C}(S^*\Omega).$$

The sheaf $\mathcal{C}$ is flabby as well as the sheaf $\mathcal{B}$. (See Sato-Kashiwara [3], Sato-Kawai-Kashiwara [4].)

Let $\mathcal{H}^s_{\text{loc}}$ be the sheaf of distributions in the local Sobolev space $H^s_{\text{loc}}(\Omega)$. In this note we decompose the sheaf $\mathcal{H}^s_{\text{loc}}/\mathcal{E}$ to obtain a sheaf $\mathcal{M}^s$ over the cosphere bundle $S^*\Omega$ such that the following isomorphisms

$$H^s_{\text{loc}}(\Omega)/\mathcal{E}(\Omega)\cong\pi_*\mathcal{M}^s(\Omega)\cong\mathcal{M}^s(S^*\Omega)$$

hold. This sheaf $\mathcal{M}^s$ is soft.

The supports of sections of $\mathcal{M}^s$ are closed subsets of the cosphere bundle $S^*\Omega$. These correspond to what is called "singular supports $S-S$" in the theory of the sheaf $\mathcal{C}$. Their projections to the base space Ω coincide with the classical singular supports of distributions. Our definition of the sheaf $\mathcal{M}^s$ is essentially the same as announced in Hörmander's paper [1]. And the wave front sets introduced by him in the case of $\mathcal{D}'/\mathcal{E}$ are nothing but the supports of the sections of our sheaf $\mathcal{M}^{-\infty}$.

Let ω be an open set in Ω and σ be an open set in the unit sphere S^{n-1} in $\boldsymbol{R}^n$.

We shall introduce linear spaces as the following.

$H^{s,\infty}_{\text{loc}}(\omega\times\sigma)=\{u\in H^s_{\text{loc}}(\omega)$; for any compact sets $K\subset\omega\subset\Omega$ and $\kappa\subset\sigma\subset S^{n-1}$, there exists a function $\phi_K\in C^\infty_0(\omega)$ such that (i) $\phi_K\geqq 0$ and $\phi_K\equiv 1$ near K and (ii) for any positive integer N, $|\widehat{\phi u}(\xi)|\leqq C_N/(1+|\xi|)^N$ so long as the direction of ξ lies in κ.}

Here $\hat{v}$ stands for the Fourier transform of v.

Lemma 1. *Let $\sigma_{\xi_0} \subset S^{n-1}$ be a neighborhood of the direction of ξ_0. Assume that the rapidly decreasing estimate for $u \in \mathcal{E}'(\omega)$*

(*) $\quad |\hat{u}(\xi)| \leqq C_N/(1+|\xi|)^N$ *for any positive integer N*

holds so long as the direction of ξ lies in σ_{ξ_0}. Then for any $\phi \in C_0^\infty(\omega)$ the estimate (*) of $\widehat{\phi u}$ holds so long as the direction of ξ lies in a smaller neighborhood thereof.

The conditions in the above definition can be localized.

Lemma 2. *$H^{s,\infty}_{loc}(\omega \times \sigma) = \{u \in H^s_{loc}(\omega)$; for any $(x_0, \xi_0) \in \omega \times \sigma$ there exist a function $\phi \in C_0^\infty(\omega)$ such that (i) $\phi(x_0) \neq 0$ and (ii) $|\widehat{\phi u}(\xi)| \leqq C_N/(1+|\xi|)^N$ for any integer $N \geqq 0$ so long as the direction of ξ lies in σ_{ξ_0}.*

When $s = -\infty$, $H^{s,\infty}_{loc}(\omega \times \sigma)$ is equal to the space $\mathcal{D}'_{\complement[\sigma]}(\omega)$ of Hörmander where $[\sigma]$ is the open cone spanned by the origin and σ.

We define $M^s(\omega \times \sigma)$ as the quotient space $H^s_{loc}(\omega)/H^{s,\infty}_{loc}(\omega \times \sigma)$. The correspondence $M^s : \omega \times \sigma \mapsto M^s(\omega \times \sigma)$ defines a presheaf. The sheaf associated with M^s is denoted by $\mathcal{M}^s$. Our results are following theorems.

Theorem 1. *The sheaves $\mathcal{H}^s_{loc}/\mathcal{E}$ and $\pi_* \mathcal{M}^s$ are isomorphic. Moreover the global sections $H^s_{loc}(\Omega)/\mathcal{E}(\Omega)$ and $\mathcal{M}^s(\Omega \times S^{n-1})$ are isomorphic.*

Theorem 2. *The sheaf $\mathcal{M}^s$ is soft.*

Outline of proofs. We need some notations. We denote a finite covering of S^{n-1} by $\mathcal{S}$. We put $Z(\omega \times \mathcal{S}; M^s) = \{(f_\sigma)_{\sigma \in \mathcal{S}}; f_\sigma \in M^s(\omega \times \sigma)$ and $f_\sigma = f_{\sigma'}$ on $\omega \times (\sigma \cap \sigma')\}$.

Lemma 3. *Let ω, ω' and ω'' be neighborhoods of x. Assume that ω' and ω'' are relatively compact in ω and ω' respectively. Let $\mathcal{S}'$ be a finite refinement of the covering $\mathcal{S}$. Then there exist mappings, shown by broken arrows, which make the diagram commutative.*

$$\begin{array}{ccc} Z(\omega \times \mathcal{S}, M^s) & \dashrightarrow & H^s_{loc}(\omega')/\mathcal{E}(\omega') \\ {\scriptstyle①}\downarrow & {\scriptstyle②}\swarrow \ \searrow & \downarrow{\scriptstyle③} \\ Z(\omega' \times \mathcal{S}', M^s) & \dashrightarrow & H^s_{loc}(\omega'')/\mathcal{E}(\omega''). \end{array}$$

Here the mappings ①, ② and ③ are defined by restriction.

With this lemma we are able to go to

Proof of Theorem 1. The stalk of $\pi_* \mathcal{M}^s$ at x $(\pi_* \mathcal{M}^s)_x = \varinjlim_{\omega \ni x} (\pi_* \mathcal{M}^s)(\omega) = \varinjlim_{\omega \ni x} \mathcal{M}^s(\omega \times S^{n-1}) = \varinjlim_{\omega, \mathcal{S}} Z(\omega \times \mathcal{S}; M^s)$. Lemma 3 shows that the right hand side is isomorphic to $\varinjlim_{\omega \ni x} H^s_{loc}(\omega)/\mathcal{E}(\omega) = (\mathcal{H}^s_{loc}/\mathcal{E})_x$.

Hence $\mathcal{H}^s_{\mathrm{loc}}/\mathcal{E}\cong\pi_*\mathcal{M}^s$. This gives an exact sequence:

$$0\to\mathcal{E}\to\mathcal{H}^s_{\mathrm{loc}}\to\pi_*\mathcal{M}^s\to 0.$$

Since $\mathcal{E}$ is a fine sheaf, this induces an exact sequence of global sections:

$$0\to\mathcal{E}(\Omega)\to H^s_{\mathrm{loc}}(\Omega)\to(\pi_*\mathcal{M}^s)(\Omega)\to 0.$$

Therefore $H^s_{\mathrm{loc}}(\Omega)/\mathcal{E}(\Omega)\cong\mathcal{M}^s(\Omega\times S^{n-1})$. Theorem 1 is thus proved.

Proof of Lemma 3. Let $(f_\sigma)_{\sigma\in\mathcal{S}}$ be an element in $Z(\omega\times\mathcal{S},M^s)$. Each f_σ belonging to $M^s(\omega\times\sigma)$ is represented by $u_{\omega\sigma}\in H^s_{\mathrm{loc}}(\omega)$. We define $v_{\omega'}=\mathcal{F}^{-1}\Big[\sum_{\sigma\in\mathcal{S}}\alpha(\xi)\beta_\sigma(\xi/|\xi|)\widehat{\phi_{\omega'}u_{\omega\sigma}}(\xi)\Big]$. Here $\mathcal{F}^{-1}$ denotes the inverse Fourier transformation. $\alpha(t)$ is such a C^∞-function as $\alpha(t)\equiv 0$ near 0 and $\alpha(t)\equiv 1$ outside $|t|\leqq 1$. The collection $\{\beta_\sigma(\xi)\}_{\sigma\in\mathcal{S}}$ is a partition of unity subordinate to the covering S of S^{n-1}. And $\phi_{\omega'}$ is the smooth function stated in the definition of $H^{s,\infty}_{\mathrm{loc}}(\omega\times\sigma)$. This mapping $(u_{\omega\sigma})_\sigma\mapsto v_{\omega'}$ is what we want. The ambiguity caused by selections of α, $\{\beta_\sigma\}$ and $\phi_{\omega'}$ is absorbed in $\mathcal{E}(\omega)$. Commutativity is a consequence of following ones.

$$\begin{array}{ccccc}
Z(\omega\times\mathcal{S},M^s) & \to & H^s_{\mathrm{loc}}(\omega')/\mathcal{E}(\omega'), & & Z(\omega\times\mathcal{S},M^s)\to H^s_{\mathrm{loc}}(\omega')/\mathcal{E}(\omega') \\
\downarrow & \swarrow & & & \downarrow \quad \nearrow \\
Z(\omega'\times\mathcal{S},M^s) & & & & Z(\omega\times\mathcal{S}',M^s)\ .
\end{array}$$

Let $(u_{\omega\sigma})_{\sigma\in\mathcal{S}}$ be an element in $Z(\omega\times\mathcal{S},M^s)$. Let ϕ be any smooth function in $C_0^\infty(\omega')$. Then

$$\begin{aligned}
&\phi\Big(\mathcal{F}^{-1}\sum_{\tau\in\mathcal{S}}\beta_\tau\widehat{\phi_{\omega'}u_{\omega\tau}}-u_{\omega\sigma}\Big)\\
&\qquad=\phi\Big(\mathcal{F}^{-1}\sum_{\tau\in\mathcal{S}}\beta_\tau\widehat{\phi_{\omega'}(u_{\omega\tau}-u_{\omega\sigma})}\Big)\qquad \mathrm{mod.}\ \mathcal{E}(\omega),
\end{aligned}$$

and its Fourier transformation is rapidly decreasing so long as the direction of ξ lies in σ by Lemma 1. Therefore the first diagram is commutative. We can verify similarly that the second diagram is also commutative. Lemma 3 is thus proved.

Proof of Theorem 2. We can make use of the partition of unity not only on the base space Ω, but also on the fiber S^{n-1} as we stated in the proof of Lemma 3. This procedure is not difficult but the details are omitted here.

Remark 1. Arguments as to the change of the variables (see Hörmander [2]) show that $\Omega\times S^{n-1}$ should be regarded as the cosphere bundle $S^*\Omega$.

Remark 2. It is not clear for us whether the sheaf $\mathcal{D}'/\mathcal{A}$ can be decomposed by the method of Fourier transformation (cf. Hörmander [2]).

References

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