86. *Oscillatory Integrals of Symbols of Pseudo-Differential Operators on R^n and Operators of Fredholm Type*

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Introduction. In this paper we shall introduce the oscillatory integral of the form $O_s-\iint e^{-ix\cdot\xi}p(\xi, x)dxd\xi$ for a C^∞-function $p(\xi, x)$ of class $\mathcal{A}$ (defined in Section 1), and by using this integral study the algebra of pseudo-differential operators of class $S^m_{\lambda,\rho,\delta}$, $0\leqq\delta\leqq\rho\leqq 1, \delta<1$, whose basic weight function $\lambda=\lambda(x, \xi)$ varies even in x and may increase in polynomial order.*) The Friedrichs part P_F of the operator P of class $S^m_{\lambda,\rho,\delta}$ will be defined as in Kumano-go [6]. Then, the L^2-boundedness for the operator P of class $S^0_{\lambda,\rho,\delta}$ for $\delta<\rho$, can be proved by using P_F and the Calderon-Vaillancourt theorem in [1]. We have to note that all the results obtained there hold even for operator-valued symbols as in Grushin [3].

Next we shall give a sufficient condition in order that an operator of class $S^m_{\lambda,\rho,\delta}$ is Fredholm type. Finally we shall derive a similar inequality to that of Grushin [3] for an operator with polynomial coefficients and with mixed homogeneity in (x, ξ), and give a theorem on hypoellipticity at the origin.

All the theorems are stated without proofs and the detailed description will be published elsewhere.

§1. Oscillatory integrals.

Definition 1.1. We say that a C^∞-function $p(\xi, x)$ in $R^{2n}_{\xi,x}$ belongs to a class $\mathcal{A}^m_\delta$, $-\infty<m<\infty, 0\leqq\delta<1$, when for any multi-index α, β we have

$$|p^{(\alpha)}_{(\beta)}(\xi, x)|\leqq C_{\alpha,\beta}\langle x\rangle^{l_\beta}\langle\xi\rangle^{m+\delta|\beta|} \tag{1.1}$$

for constants $C_{\alpha,\beta}$ and l_β, where $p^{(\alpha)}_{(\beta)}=\partial^\alpha_\xi D^\beta_x p$, $D_{x_j}=-i\partial/\partial x_j$, $\partial_{\xi_j}=\partial/\partial\xi_j$, $j=1, \cdots, n$, $\langle x\rangle=\sqrt{1+|x|^2}$, $\langle\xi\rangle=\sqrt{1+|\xi|^2}$. We set

$$\mathcal{A}=\bigcup_{0\leqq\delta<1}\ \bigcup_{-\infty<m<\infty}\mathcal{A}^m_\delta$$

(cf. [8]).

Definition 1.2. For a $p(\xi, x)\in\mathcal{A}$ we define the oscillatory integral $O_s[p]$ by

*) R. Beals and C. Fefferman have reported to us that they discovered a new class $S^{M,m}_{\Phi,\phi}$ of pseudo-differential operators, which is defined by basic weight functions $\Phi(x,\xi)$ and $\phi(x,\xi)$ depending on x and ξ, and covers Hörmander's class $S^m_{\rho,\delta}$ in [4].

$$
(1.2)\qquad \begin{aligned} O_s[p] &\equiv O_s-\iint e^{-ix\cdot\xi}p(\xi, x)dxd\xi \\ &=\lim_{\varepsilon\to 0}\iint e^{-ix\cdot\xi}\chi_\varepsilon(\xi, x)p(\xi, x)dxd\xi, \end{aligned}
$$

where $d\xi=(2\pi)^{-n}d\xi$, $x\cdot\xi=x_1\xi_1+\cdots+x_n\xi_n$ and $\chi_\varepsilon(\xi, x)=\chi(\varepsilon\xi, \varepsilon x)$, $0<\varepsilon\leqq 1$, for a $\chi(\xi, x)\in\mathcal{S}$ (the class of rapidly decreasing functions of Schwartz) in $R^{2n}_{\xi,x}$ such that $\chi(0,0)=1$.

Lemma 1.3. i) *For a* $p(\xi, x)\in\mathcal{A}^m_\delta$ *we choose positive integers* l *and* l' *such that* $-2l(1-\delta)+m<-n$ *and* $-2l'+\mathrm{Max}_{|\beta|\leqq 2l}\{l_\beta\}<-n$. *Then we can write* $O_s[p]$ *as*

$$
O_s[p]=\iint e^{-ix\cdot\xi}\langle x\rangle^{-2l'}\langle D_\xi\rangle^{2l'}\{\langle\xi\rangle^{-2l}\langle D_x\rangle^{2l}p(\xi, x)\}dxd\xi,
$$

and we have for $l_0=2(l+l')$ $|O_s[p]|\leqq C|p|^{(m)}_{l_0}$ *with a constant* C *independent of* $p(\xi, x)$, *where* $|p|^{(m)}_{l_0}=\mathrm{Max}_{|\alpha+\beta|\leqq l_0}\inf\{C_{\alpha,\beta}$ *of* (1.1)$\}$.

ii) *For* $p_j(\xi, x)\in\mathcal{A}$, $j=1,2$, *we have*

$$
O_s[\partial_{\xi_j}p_1\cdot p_2]=O_s[p_1(ix_jp_2-\partial_{\xi_j}p_2)],
$$
$$
O_s[\partial_{x_j}p_1\cdot p_2]=O_s[p_1(i\xi_jp_2-\partial_{x_j}p_2)].
$$

§ 2. Class $S^m_{\lambda,\rho,\delta}$ of pseudo-differential operators.

Definition 2.1. We say that a C^∞-function $\lambda(x,\xi)$ is a basic weight function when $\lambda(x,\xi)$ satisfies for constants $A_0, A_{\alpha,\beta}$ and A_1

$$
(2.1)\qquad 1\leqq\lambda(x,\xi)\leqq A_0\langle x\rangle^{\tau_0}\langle\xi\rangle \qquad (\tau_0\geqq 0),
$$
$$
(2.2)\qquad |\lambda^{(\alpha)}_{(\beta)}(x,\xi)|\leqq A_{\alpha,\beta}\lambda(x,\xi)^{1+\delta|\beta|-|\alpha|} \qquad (0\leqq\delta<1),
$$
$$
(2.3)\qquad \lambda(x+y,\xi)\leqq A_1\langle y\rangle^{\tau_1}\lambda(x,\xi) \qquad (\tau_1\geqq 0).
$$

Definition 2.2. We say that a C^∞-function $p(x,\xi)$ belongs to a class $S^m_{\lambda,\rho,\delta}$, $0\leqq\delta\leqq\rho\leqq 1$, when

$$
(2.4)\qquad |p^{(\alpha)}_{(\beta)}(x,\xi)|\leqq C_{\alpha,\beta}\lambda(x,\xi)^{m+\delta|\beta|-\rho|\alpha|} \qquad \text{(cf. [4])},
$$

and the pseudo-differential operator $P=p(X,D_x)$ is defined by

$$
(2.5)\qquad Pu(x)=\int e^{ix\cdot\xi}p(x,\xi)\hat{u}(\xi)d\xi \qquad \text{for } u\in\mathcal{S},
$$

where $\hat{u}(\xi)=\int e^{-ix'\cdot\xi}u(x')dx'$ is the Fourier transform of $u\in\mathcal{S}$.

Remark 1°. $S^m_{\lambda,\rho,\delta}$ makes a Fréchet space by semi-norms $|p|^{(m)}_l$, $l=0,1,2,\cdots$ defined by

$$
|p|^{(m)}_l=\underset{|\alpha+\beta|\leqq l}{\mathrm{Max}}\ \sup_{(x,\xi)}\{|p^{(\alpha)}_{(\beta)}(x,\xi)|\lambda(x,\xi)^{-m-\delta|\beta|+\rho|\alpha|}\}.
$$

2°. It is easy to see that P is a continuous map of $\mathcal{S}$ into $\mathcal{S}$, so that from Theorem 2.5 P can be extended uniquely to the map of $\mathcal{S}'$ into $\mathcal{S}'$ by $(Pu, v)=(u, P^{(*)}v)$ for $u\in\mathcal{S}'$, $v\in\mathcal{S}$.

Theorem 2.3. *Let* $P_j=p_j(X,D_x)\in S^{m_j}_{\lambda,\rho,\delta}$, $j=1,2$. *Then,* $P=P_1P_2\in S^{m_1+m_2}_{\lambda,\rho,\delta}$ *and setting*

$$
\begin{cases} p_\alpha(x,\xi)=p_1^{(\alpha)}(x,\xi)p_{2(\alpha)}(x,\xi) \qquad (\in S^{m_1+m_2-(\rho-\delta)|\alpha|}_{\lambda,\rho,\delta}), \\ r_{\gamma,\theta}(x,\xi)=O_s-\iint e^{-iy\cdot\eta}p_1^{(\gamma)}(x,\xi+\theta\eta)p_{2(\gamma)}(x+y,\xi)dyd\eta \end{cases}
$$

we have for any integer $N>0$

$$(2.6)\quad \sigma(P)(x,\xi)=\sum_{|\alpha|<N}\frac{1}{\alpha!}p_\alpha(x,\xi)+N\sum_{|\gamma|=N}\int_0^1\frac{(1-\theta)^{N-1}}{\gamma!}r_{\gamma,\theta}(x,\xi)d\theta.$$

The set $\{r_{\gamma,\theta}(x,\xi)\}_{|\theta|\leq 1}$ *is bounded in* $S^{m_1+m_2-(\rho-\delta)|\gamma|}_{\lambda,\rho,\delta}$.

Lemma 2.4. *Define a class* $S^{m,m'}_{\lambda,\rho,\delta}$ *of double symbols* $p(\xi,x',\xi')$ *by*

$$|p^{(\alpha,\alpha')}_{(\beta)}(\xi,x',\xi')|\leqq C_{\alpha,\alpha',\beta}\lambda(x',\xi)^{m-\rho|\alpha|}(\lambda(x',\xi)+\lambda(x',\xi'))^{\delta|\beta|}\lambda(x',\xi')^{m'-\rho|\alpha'|}.$$

Then, the operator $P=p(D_x,X',D_{x'})$ *defined by*

$$\widehat{Pu}(\xi)=O_s-\iint e^{-ix'\cdot(\xi-\xi')}p(\xi,x',\xi')\hat{u}(\xi')d\xi'dx'\qquad \text{for } u\in\mathcal{S}$$

belongs to $S^{m+m'}_{\lambda,\rho,\delta}$, *and setting*

$$\begin{cases} p_\alpha(x,\xi)=p^{(\alpha,0)}_{(\alpha)}(\xi,x,\xi) & (\in S^{m+m'-(\rho-\delta)|\alpha|}_{\lambda,\rho,\delta}),\\ r_{\gamma,\theta}(x,\xi)=O_s-\iint e^{-iy\cdot\eta}p^{(\gamma,0)}_{(\gamma)}(\xi+\theta\eta,x+y,\xi)dyd\eta \end{cases}$$

we can write $\sigma(P)(x,\xi)$ *in the form* (2.6) *for any* $N>0$. *The set* $\{r_{\gamma,\theta}(x,\xi)\}_{|\theta|\leq 1}$ *is bounded in* $S^{m+m'-(\rho-\delta)|\gamma|}_{\lambda,\rho,\delta}$.

Theorem 2.5. *For* $P=p(X,D_x)\in S^m_{\lambda,\rho,\delta}$ *the operator* $P^{(*)}$ *defined by* $(Pu,v)=(u,P^{(*)}v)$ *for* $u,v\in\mathcal{S}$ *belongs to* $S^m_{\lambda,\rho,\delta}$, *and setting*

$$\begin{cases} p^{(*)}_\alpha(x,\xi)=(-1)^{|\alpha|}\overline{p^{(\alpha)}_{(\alpha)}(x,\xi)} & (\in S^{m-(\rho-\delta)|\alpha|}_{\lambda,\rho,\delta}),\\ r^{(*)}_{\gamma,\theta}(x,\xi)=O_s-\iint e^{iy\cdot\eta}(-1)^{|\gamma|}\overline{p^{(\gamma)}_{(\gamma)}(x+y,\xi+\theta\eta)}dyd\eta \end{cases}$$

we have for any $N>0$

$$\sigma(P^{(*)})(x,\xi)=\sum_{|\alpha|<N}\frac{1}{\alpha!}p^{(*)}_\alpha(x,\xi)+N\sum_{|\gamma|=N}\int_0^1\frac{(1-\theta)^{N-1}}{\gamma!}r^{(*)}_{\gamma,\theta}(x,\xi)d\theta.$$

The set $\{r^{(*)}_{\gamma,\theta}(x,\xi)\}_{|\theta|\leq 1}$ *is bounded in* $S^{m-(\rho-\delta)|\gamma|}_{\lambda,\rho,\delta}$.

Let $q(\sigma)$ be a C^∞- and even-function such that $\int q(\sigma)^2 d\sigma=1$ and $\operatorname{supp} q\subset\{\sigma\in R^n\,;\,|\sigma|\leqq 1\}$ and set

$$F(x,\xi\,;\zeta)=\lambda(x,\xi)^{-n\tau/2}q((\zeta-\xi)/\lambda(x,\xi)^\tau)\qquad\text{for } \tau=(\rho+\delta)/2.$$

Theorem 2.6. *For* $P=p(X,D_x)\in S^m_{\lambda,\rho,\delta}(\delta<\rho)$ *define the Friedrichs part* $P_F=p_F(D_x,X',D_{x'})$ *by*

$$p_F(\xi,x',\xi')=\int F(x',\xi\,;\zeta)p(x',\zeta)F(x',\xi'\,;\zeta)d\zeta.$$

Then, we have $P_F\in S^m_{\lambda,\rho,\delta}$ *and* $P-P_F\in S^{m-(\rho-\delta)}_{\lambda,\rho,\delta}$, *and*

$$\sigma(P_F)(x,\xi)\sim p(x,\xi)+\sum_{\alpha,\beta,\gamma}\Psi_{\alpha,\beta,\gamma}(x,\xi)p^{(\alpha)}_{(\beta)}(x,\xi),$$

where $\Psi_{\alpha,\beta,\gamma}(x,\xi)\in S^{\tau(|\alpha|-|\beta|)-(\rho-\delta)|\gamma|/2}_{\lambda,1,\delta}$ *and the summation is taken over* (α,β,γ) *such that* $-(\rho-\delta)|\alpha+\beta+\gamma|/2\leqq-(\rho-\delta)$, *i.e.,* $|\alpha+\beta+\gamma|\geqq 2$. *Moreover, if* $p(x,\xi)$ *is real valued and non-negative, we have*

$$(P_Fu,v)=(u,P_Fv)\quad\textit{and}\quad (P_Fu,u)\geqq 0\qquad \textit{for } u,v\in\mathcal{S}.$$

Theorem 2.7. *Let* $P=p(X,D_x)\in S^0_{\lambda,\rho,\delta}(\delta<\rho)$. *Then, we have for some* l *and a constant* C

$$\|Pu\|_{L^2}\leqq C|p|^{(0)}_l\|u\|_{L^2}\qquad \textit{for } u\in L^2(R^n).$$

§3. Operators of Fredholm type. In what follows we assume that

$$(3.1)\qquad c_0\langle\xi\rangle^{a_0}\leqq\lambda(x,\xi)\qquad\text{for some } 0<a_0\leqq 1,\ 0<c_0.$$

Consider $P=p(X,D_x)\in S^m_{\lambda,\rho,\delta}$ as the closed operator of $L^2=L^2(R^n)$ into itself with the domain $\mathcal{D}(P)=\{u\in L^2\,;\,Pu\in L^2\}$.

We say that $p(x,\xi)\in S^m_{\lambda,\rho,\delta}$ is slowly varying if we have (2.4) for a bounded function $C_{\alpha,\beta}(x)$ such that $C_{\alpha,\beta}(x)\to 0$ as $|x|\to\infty$ for $\beta\neq 0$ (cf. [2]). Then we have

Theorem 3.1. *Let $P=p(X,D_x)\in S^m_{\lambda,\rho,\delta}$ for $m\geqq 0$ and $\delta<\rho$. Suppose that $p(x,\xi)$ is slowly varying and satisfies conditions:*

$$\begin{cases}|p^{(\alpha)}_{(\beta)}(x,\xi)p(x,\xi)^{-1}|\leqq C'_{\alpha,\beta}(x)\lambda(x,\xi)^{\delta|\beta|-\rho|\alpha|}\\ |p(x,\xi)|\geqq C_0\lambda(x,\xi)^{m\tau}\qquad 0<C_0,\ 0\leqq\tau\leqq 1)\end{cases}$$

for large $|x|+|\xi|$, where $C'_{\alpha,\beta}(x)$ are bounded functions such that $C'_{\alpha,\beta}(x)\to 0$ as $|x|\to\infty$ for $\beta\neq 0$. Then, P is Fredholm type in L^2, and there exist parametrices Q and Q' in $S^{-m\tau}_{\lambda,\rho,\delta}$ such that

$$QP=I+K\quad\text{and } Q'P^{(*)}=I+K', \tag{3.2}$$

where K and K' belong to $S^{-\infty}_{\lambda,\rho,\delta}$ and are compact in L^2. (cf. [4], [7], [9])

Remark. When $\lambda(x,\xi)\to\infty$ as $|x|\to\infty$, symbols of class $S^m_{\lambda,\rho,\delta}(\delta<\rho)$ are always slowly varying in $S^m_{\lambda,\rho,\delta'}$, for $\delta<\delta'<\rho$.

§4. Examples. Let $\mathfrak{m}=(m_1,\cdots,m_n,m'_1,\cdots,m'_k)$ be a multi-index of positive integers m_j and m'_l. Consider an operator $L(x,\tilde{y},D_x,D_y)$ in $R^n_x\times R^k_y$ with polynomial coefficients and of the form

$$L(x,\tilde{y},\xi,\eta)=\sum_{|\alpha:\mathfrak{m}|\leqq 1}a_{\alpha,\gamma}(x,\tilde{y})^\gamma(\xi,\eta)^\alpha, \tag{4.1}$$

and set

$$L_0(x,\tilde{y},\xi,\eta)=\sum_{|\alpha:\mathfrak{m}|=1}a_{\alpha,\gamma}(x,\tilde{y})^\gamma(\xi,\eta)^\alpha, \tag{4.2}$$

where $y=(\tilde{y},\tilde{\tilde{y}}),\ \tilde{y}=(y_1,\cdots,y_s),\ \tilde{\tilde{y}}=(y_{s+1},\cdots,y_k)$ for $s\leqq k$,

$$\alpha=(\alpha_1,\cdots,\alpha_n,\alpha'_1,\cdots,\alpha'_k),\ \gamma=(\gamma_1,\cdots,\gamma_n,\gamma'_1,\cdots,\gamma'_s,0,\cdots,0),$$
$$|\alpha:\mathfrak{m}|=\alpha_1/m_1+\cdots+\alpha_n/m_n+\alpha'_1/m'_1+\cdots+\alpha'_k/m'_k,$$
$$(x,\tilde{y})^\gamma=x_1^{\gamma_1}\cdots x_n^{\gamma_n}y_1^{\gamma'_1}\cdots y_s^{\gamma'_s},\ (\xi,\eta)^\alpha=\xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n}\eta_1^{\alpha'_1}\cdots\eta_k^{\alpha'_k}.$$

Now setting $m=\mathrm{Max}\,\{m_j,m'_l\}$ we assume that there exist two real vectors $\rho=(\rho_1,\cdots,\rho_n,\rho'_1,\cdots,\rho'_k),\ \sigma=(\sigma_1,\cdots,\sigma_n,\sigma'_1,\cdots,\sigma'_s,0,\cdots,0)$ such that

$$\begin{aligned}&\text{(i)}\quad \rho_j=\sigma_j=m/m_j,\qquad j=1,\cdots,n,\\ &\text{(ii)}\quad \rho'_j>\sigma'_j\geqq 0,\quad \rho'_jm'_j\geqq m,\quad j=1,\cdots,k,\end{aligned} \tag{4.3}$$

and

$$L(t^{-\sigma}(x,\tilde{y}),t^\rho(\xi,\eta))=t^mL(x,\tilde{y},\xi,\eta)\qquad\text{for } t>0 \tag{4.4}$$

where $t^{-\sigma}(x,\tilde{y})=(t^{-\sigma_1}x_1,\cdots,t^{-\sigma_n}x_n,t^{-\sigma'_1}y_1,\cdots,t^{-\sigma'_s}y_s)$,

$$t^\rho(\xi,\eta)=(t^{\rho_1}\xi_1,\cdots,t^{\rho_n}\xi_n,t^{\rho'_1}\eta_1,\cdots,t^{\rho'_k}\eta_k),$$

and assume that

$$L_0(x,\tilde{y},\xi,\eta)\neq 0\quad\text{for}\quad |x|+|\tilde{y}|\neq 0\quad\text{and}\quad(\xi,\eta)\neq 0, \tag{4.5}$$

which means that $L(x,\tilde{y},\xi,\eta)$ is semi-elliptic for $|x|+|\tilde{y}|\neq 0$. Then from (4.3)–(4.5) we have for a constant $C>0$

$$C^{-1}|L_0(x,\tilde{y},\xi,\eta)|\leqq\Big\{\sum_{j=1}^n|\xi_j|^{m_j}+\sum_{j=1}^n|x,\tilde{y}|_\sigma^{(\rho'_jm'_j-m)}|\eta_j|^{m'_j}\Big\}\leqq C\,|L_0(x,\tilde{y},\xi,\eta)|,$$

where

$$|x, \tilde{y}|_\sigma = \left\{\sum_{j=1}^{n} |x_j|^{1/\sigma_j} + \sum_{j=1}^{s} |y_j|^{1/\sigma'_j}\right\}.$$

Using this we get a basic weight function $\lambda_h(x, \xi)(|\eta|=1)$ with parameter $h=(\tilde{y}, \eta)$ by $\lambda_h(x, \xi)=\{1+|L(x, \tilde{y}, \xi, \eta)|^2\}^{1/2m}$ $(|\eta|=1)$ for $\delta=0$ and $a_0 = \mathrm{Min}_{1\leq j\leq n}\{m_j/m\}$. Setting $p_h(x, \xi)=L(x, \tilde{y}, \xi, \eta)$ we can check that $p_h(x, \xi) \in S^m_{\lambda_h,1,0}$ and satisfies the conditions of Theorem 3.1 for $\tau=1$ and for large $|x|+|\tilde{y}|+|\xi|$. Moreover, we can replace $C_{\alpha,\beta}(x)$ by bounded functions $C_{\alpha,\beta}(x, \tilde{y})$ such that

(4.6) $\qquad C_{\alpha,\beta}(x, \tilde{y})\to 0$ as $|x|+|\tilde{y}|\to\infty$ $\qquad$ for $\beta\neq 0$.

Then we have for a compact operator $K_h(X, D_x)$

(4.7) $\quad \|u\|_{L^2_x} \leqq C\|L(X, \tilde{y}, D_x, \eta)u\|_{L^2_x} + \|K_h(X, D_x)u\|_{L^2_x}$ $\qquad$ for $u \in \mathcal{S}_x$.

Moreover, if we add an assumption that the equation $L(X, \tilde{y}, D_x, \eta)u(x) = 0$ $(|\eta|=1)$ has no non-trivial solution in $\mathcal{S}_x$, then by using (4.6) and the relation: $t^m\|L(X, \tilde{y}, D_x, \eta)u\|_{L^2_x} = \|L(t^{-\sigma}(X, \tilde{y}), t^\rho(D_x, \eta))u\|_{L^2_x} = t^{(\Sigma_{j=1}^n \sigma_j)/2}$ $\|L(X, t^{-\sigma'}\tilde{y}, D_x, t^{\sigma'}\eta)v\|_{L^2_x}$ for $v(x)=u(t^{\sigma_1}x_1, \cdots, t^{\sigma_n}x_n)$ we have

(4.8) $\quad |\eta|^m_{\rho'}\|u\|_{L^2_x} \leqq C'\|L(X, \tilde{y}, D_x, \eta)u\|_{L^2_x}$ $\qquad$ for $u \in \mathcal{S}_x$ and $\eta \in R^k$,

where $\sigma'=(\sigma'_1, \cdots, \sigma'_s)$, $\rho'=(\rho'_1, \cdots, \rho'_k)$ and $|\eta|_{\rho'}=\sum_{j=1}^k |\eta_j|^{1/\rho'_j}$. Finally we have

Theorem 4.1. *The operator $L(x, \tilde{y}, D_x, D_y)$ which satisfies* (4.4) *and* (4.5) *is hypoelliptic at the origin, if* (*and only if when $\tilde{y}$ does not appear*) *$L(X, \tilde{y}, D_x, \eta)u=0$ has no non-trivial solution in $\mathcal{S}_x$ for $|\eta|=1$ and $\mathrm{Max}_{1\leq j\leq k}\{\sigma'_j\} < \mathrm{Min}_{1\leq j,l\leq k}\{m'_j\rho'_j/m'_l\}$.*

Example 1°. $L_\pm(x, D_x, D_y)=D_x \pm ixD_y^2$ in $R^1_x \times R^1_y$ (cf. [5]).

$$\mathfrak{m}=(1, 2),\quad m=2,\quad \rho_1=\sigma_1=2,\quad \rho_2=2,\quad \sigma_2=0.$$

In this case $L_+(X, D_x, \pm 1)u=0$ has no non-trivial solution in $\mathcal{S}_x$ and $L_-(X, D_x, \pm 1)u=0$ has non-trivial solution $e^{-x^2/2} \in \mathcal{S}_x$.

2°. $L_k(x, D_x, D_y)=D_x+ix^kD_y$ in $R^1_x \times R^1_y$ (cf. [10]). $\mathfrak{m}=(1, 1)$, $m=1$, $\rho_1=\sigma_1=1$, $\rho_2=k+1$, $\sigma_2=0$. In this case $L_k(X, D_x, \pm 1)u=0$ has non-trivial solution in $\mathcal{S}_x$ for even k and $L_k(X, D_x, -1)u=0$ has non-trivial solution $e^{-x^{k+1}/(k+1)}$ for odd k.

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