116. *Thin Sets in an Open Unit Disk*

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1. Introduction. The purpose of this paper is to establish the following theorem.

Theorem. *Let F be a closed subset of an open unit disk $U=\{|z|<1\}$. Suppose the circular projection $T(F)$ of F contains some countable union $\{E_n\}_{n=1}^{\infty}$ of closed intervals such that each E_n $(n=1,2,\cdots)$ is a closed interval $[a_n, b_n]$ with $0<a_n<b_n<a_{n+1}<1$ and $\lim_{n\to\infty} a_n=1$. Set $\lambda_k=\inf_{x\in E_k}\sup_{z\in F,|z|=x} k_1(z)$ $(k=1,2,\cdots)$. If $\overline{\lim_{n\to\infty}}\frac{1}{1-a_n}\sum_{k=n}^{\infty}\lambda_k(b_k-a_k)(1-a_kb_k)>0$, then F is not thin at $z=1$.*

Notation and terminology. Let C be a complex plane. For a subset A of C, we denote by ∂A the boundary of A in C. Let U be an open unit disk $\{|z|<1\}$ in C in this paper. Set $T(z)=|z|$ $(z\in U)$. Then T is a continuous mapping of U into U. For a subset A of U, we say that $T(A)$ is the circular projection of A. Let a and b two points of U. Then we define the hyperbolic distance (or length) $\delta(a,b)$ of a and b by $\delta(a,b)=\left|\frac{a-b}{1-\bar{a}b}\right|$. For a subset A of U, the hyperbolic diameter $\delta(A)$ of A is defined by $\delta(A)=\sup_{a,b\in A}\delta(a,b)$.

We shall use the same notations as in [3], for instance, $C_0(X)$, $\overline{H}_f^G$, $\underline{H}_f^G$, H_f^G, $\omega_a^G=\omega_a=\omega$, s_F, the Green capacity C, etc.

2. Green potentials on U. Let μ be a (positive Radon) measure on U. Set $L(f)=\int f\circ T d\mu$ for each f of $C_0(U)$. Then L is a positive linear functional on $C_0(U)$. By Riesz representation theorem, there exists a (positive Radon) measure μ^T on U such that $L(f)=\int f d\mu^T$.

The following properties are easy to see:

(i) $\int f d\mu^T=\int f(|z|)d\mu(z)$ for any non-negative Borel measurable function f on U,

(ii) $\int d\mu=\int d\mu^T$,

(iii) $S(\mu^T)=T(S_\mu)$, where S_μ is the support of μ.

Let $g(z,\zeta)=\log\left|\frac{1-\bar{z}\zeta}{z-\zeta}\right|$ denote the Green function on U with pole at $\zeta\in U$ and p^μ be a Green potential associated with a (positive Radon) measure μ on U. Since $g(-|z|,|\zeta|)\leqq g(z,\zeta)\leqq g(|z|,|\zeta|)$, we have

Lemma 1. *$p^{\mu^T}(-|z|)\leqq p^{\mu}(z)\leqq p^{\mu^T}(|z|)$ in U.*

By an argument similar to the proof of Hilfssatz 19.1 in [3] and elementary properties of capacity, we have

Lemma 2. *Let F be a K_σ set in U. Then $C(F)\geqq C(T(F))$.*

Corollary. *Let F be a K_σ set in U. If $C(F)=0$, then $C(T(F))=0$.*

3. Proposition and lemma.

Proposition. *Let F be a closed subset of U and s be a non-negative superharmonic function in U. Set $E=T(F)$. We define a function ϕ on $\partial(R-E)$ such that*

$$\phi(\zeta)=\begin{cases}\sup\limits_{z\in F,|z|=\zeta} s(z) & \zeta\in E,\\ 0 & \zeta\in\partial U.\end{cases}$$

Then $s_F(z)\geqq\overline{H}_\phi^{U-E}(-|z|)$ in $U-E$.

Proof. First suppose F is an arbitrary compact subset of U. Since s_F is a Green potential, by Frostman's theornm there exists a measure μ on F such that $s_F=p^\mu$. Set $w=p^{\mu^T}$. By Kellogg's theorem and the Corollary to Lemma 2, we see that $w\geqq\phi$ quasi everywhere[1] on E. We define a function ψ on $\partial(U-E)$ as follows $\psi=w$ on E and 0 on ∂U. Then $\psi\geqq\phi$ quasi everywhere on $\partial(U-E)$. It follows that $w\geqq w_E=H_\psi^{U-E}\geqq\overline{H}_\phi^{U-E}$ in $U-E$ (cf. [4]). On the other hand, it follows from Lemma 1 that $s_F(z)=p^\mu(z)\geqq p^{\mu^T}(-|z|)=w(-|z|)$ in U.

Secondly suppose F is an arbitrary closed set in U. Set $F_n=F\cap\left\{|z|\leqq 1-\frac{1}{n}\right\}$ and $E_n=T(F_n)$ $(n=1,2,\cdots\cdots)$. We define two functions ϕ_n and ψ_n as follows

$$\phi_n=\begin{cases}\phi & \text{on } E_n,\\ 0 & \text{on } \partial U\end{cases}$$

and

$$\psi_n=\begin{cases}\phi & \text{on } E_n\\ 0 & \text{on } \partial U\cup(E-E_n).\end{cases}$$

Then $\overline{H}_{\phi_n}^{U-E_n}\geqq\overline{H}_{\psi_n}^{U-E}$ in $U-E$ and ψ_n increases to ϕ on $\partial(U-E)$ as $n\to\infty$. On observing that $s_F(z)\geqq s_{F_n}(z)\geqq\overline{H}_{\phi_n}^{U-E_n}(-|z|)\geqq\overline{H}_{\psi_n}^{U-E}(-|z|)$ in $U-E$ and that $\overline{H}_{\psi_n}^{U-E}$ converges to $\overline{H}_\phi^{U-E}$ as $n\to\infty$ (cf. [2], [4]), we have $s_F(z)\geqq\overline{H}_\phi^{U-E}(-|z|)$ in $U-E$.

Corollary (A. Beurling [1].

$$1_F(z)\geqq\omega_{-|z|}^{U-T(F)}(T(F))\qquad\text{in } U-T(F).$$

Lemma 3 (cf. [5]). *Let G be an upper half disk $\{z\in U;\ \mathrm{Im}\, z>0\}$ and E be a Lebesque measurable set on the boundary diameter. If z is a point of G, then $\omega_z^G(E)=\frac{y}{\pi}\int_E\frac{(1-|z|^2)(1-\xi^2)}{|\xi-z|^2|1-\xi z|^2}d\xi$, where $z=x+iy$ (x,y; real numbers).*

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1) See p. 30 in [3].

Corollary. *Let E and H be two intervals $[\alpha,\beta]$ and $[a,1)$ respectively* $\left(\frac{1}{2}\leqq a\leqq\alpha<\beta<1\right)$. *Then* $\omega_0^{U-H}(E)\geqq\dfrac{(\beta-\alpha)(1-\alpha\beta)}{512\pi(1-a)}$.

Proof. We map $U-H$ onto an upper half disk $G=\{z\in U\,;\,\mathrm{Im}\,z>0\}$ by $S(z)=\sqrt{\dfrac{z-a}{1-az}}$. Then $S(0)=\sqrt{a}i$. By Lemma 1, we have

$$\begin{aligned}\omega_0^{U-H}(E)&=\frac{2\sqrt{a}}{\pi}\int_{S(\alpha)}^{S(\beta)}\frac{(1-a)(1-\xi^2)}{(\xi^2+a)(1+\xi^2a)}d\xi\\&=\frac{2}{\pi}\left[\tan^{-1}\frac{(1-a)\xi}{\sqrt{a}\,(1+\xi^2)}\right]_{S(\alpha)}^{S(\beta)}>\frac{1}{2\pi}\cdot\frac{1-a}{\sqrt{a}}\left(\frac{S(\beta)}{1+S(\beta)^2}-\frac{S(\alpha)}{1+S(\alpha)^2}\right)^{2)}\\&>\frac{1}{32}\cdot\frac{1-a}{\pi\sqrt{a}}\,\frac{(1-a^2)^3}{(1-\alpha a)^2(1-\beta a)^2}\cdot(\beta-\alpha)(1-\alpha\beta)\\&>\frac{1}{512\pi}\cdot\frac{(\beta-\alpha)(1-\alpha\beta)}{1-a}.\end{aligned}$$

4. Proof of Theorem. Let $k_{e^{i\theta}}(z)=\dfrac{1-|z|^2}{|1-ze^{-i\theta}|^2}$ be the Martin kernel on U with pole at $e^{i\theta}\in\partial U$ (cf. [3]). We say that a closed subset F of U is thin at a point $e^{i\theta}\in\partial U$ if $(k_{e^{i\theta}})_F\not\equiv k_{e^{i\theta}}$. If $F_0\subset F$ and F_0 is not thin at $e^{i\theta}$, then F is not thin at $e^{i\theta}$.

By a brief consideration, we have

Lemma 4. *Let $\{K_n\}_{n=1}^{\infty}$ be a sequence of compact subsets of U with* $\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}K_k=\emptyset$. *Set* $F_n=\bigcup_{k=n}^{\infty}K_k$ *and* $F=\bigcup_{n=1}^{\infty}K_k$. *Then F is thin at $e^{i\theta}\in\partial U$, if and only if* $\lim_{n\to\infty}(k_{e^{i\theta}})_{F_n}(a)=0$ *for a point a of U.*

Proof of Theorem. Let ϕ be a function on $\partial(U-E)$ such that

$$\phi(\zeta)=\begin{cases}\sup\limits_{z\in F,\,|z|=\zeta}k_1(z) & \zeta\in E,\\ 0 & \zeta\in\partial U.\end{cases}$$

Then there exists a positive integer n_0 such that $a_n\geqq\frac{1}{2}$ for $n\geqq n_0$. Let $K_k=F\cap T^{-1}(E_k)$, $F_n=\bigcup_{k=n}^{\infty}K_k$ $(k=1,2,\cdots)$ and $F_0=\bigcup_{n=1}^{\infty}K_n$. By the Proposition and the Corollary to Lemma 3, we have

$$\begin{aligned}(k_1)_{F_n}(0)&\geqq\overline{H}_\phi^{U-E'_n}(0)\quad\left(E_{n'}=\bigcup_n^{\infty}E_k\right)\\&=\int_{E'_n}^{-}\phi(\xi)d\omega_0^{U-E'_n}(\xi)\geqq\sum_{k=n}^{\infty}\lambda_k\omega_0^{U-E'_n}(E_k)\\&\geqq\frac{1}{512\pi(1-a_n)}\sum_{k=n}^{\infty}\lambda_k(b_k-a_k)(1-a_kb_k)\qquad(n\geqq n_0),\end{aligned}$$

so that $\lim_{n\to\infty}(k_1)_{F_n}(0)>0$. By Lemma 4, we observe that F_0 is not thin at $z=1$ and $F(\supset F_0)$ is not thin at $z=1$.

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2) If $0<y<x<1$, then $\tan^{-1}x-\tan^{-1}y=\tan^{-1}\dfrac{x-y}{1+xy}>\dfrac{1}{4}(x-y)$.

Set

$$\Delta(\theta_0)=\{z \in U\,;\, |\arg(z-1)|<\theta_0, |z-1|<\cos\theta_0\} \qquad \left(0<\theta_0<\frac{\pi}{2}\right).$$

We say that such a domain is a Stolz domain whose vertex is at $z=1$. If z belongs to $\Delta(\theta_0)$, then $\dfrac{|1-z|}{1-|z|} \leqq \dfrac{2}{\cos\theta_0}$ and hence $k_1(z)=\dfrac{1-|z|^2}{|1-z|^2} \geqq \dfrac{\cos^2\theta_0}{4}\cdot\dfrac{1}{1-|z|}$. Then we infer that

Corollary 1. *Let F be a closed subset of a Stolz domain $\Delta(\theta_0)$ whose vertex is at $z=1$. Suppose the circular projection $T(F)$ of F contains some countable union $\{E_n\}_{n=1}^{\infty}$ of closed intervals such that each E_n $(n=1,2,\cdots)$ is a closed interval $[a_n, b_n]$ with $0<a_n<b_n<a_{n+1}<1$ and $\lim_{n\to\infty} a_n=1$. If*

$$\varlimsup_{n\to\infty} \frac{1}{1-a_n}\sum_{k=n}^{\infty}\frac{(b_k-a_k)(1-a_kb_k)}{1-a_k}>0,$$

then F is not thin at $z=1$.

Corollary 2. *Let K_n $(n=1,2,\cdots)$ be a closed interval $[a_n, b_n]$ such that $0<a_n<b_n<a_{n+1}<1$ and $\lim_{n\to\infty} a_n=1$. Set $F=\bigcup_{n=1}^{\infty} K_n$. If $\varlimsup_{n\to\infty} \dfrac{m(\bigcup_{k=n}^{\infty}[a_k, b_k])}{m([a_n, 1))}>0$[3], then F is not thin at $z=1$. In particular, if $\varlimsup_{n\to\infty} \delta(K_n)>0$, then F is not thin at $z=1$.*

Proof. Since

$$\begin{aligned}
&\frac{1}{1-a_n}\sum_{k=n}^{\infty}\frac{(b_k-a_k)(1-a_kb_k)}{1-a_k}\\
&\quad\geqq \frac{1}{1-a_n}\sum_{k=n}^{\infty}(b_k-a_k)\\
&\quad\geqq \frac{1}{1-a_n}\sum_{k=n}^{\infty}\frac{\delta(K_n)}{1+\delta(K_n)}(1-a_k)\\
&\quad\geqq \frac{\delta(K_n)}{1+\delta(K_n)}
\end{aligned}$$

and

$$\frac{1}{1-a_n}\sum_{k=n}^{\infty}(b_k-a_k)=\frac{m(\bigcup_{k=n}^{\infty}[a_k, b_k])}{m([a_n, 1))},$$

we obtain Corollary 2.

Example. *If we set $K_n=\left[1-\dfrac{1}{2n}, 1-\dfrac{1}{2n+1}\right]$ $(n=1,2,\cdots)$ and $F=\bigcup_{n=1}^{\infty} K_n$, then F is not thin at $z=1$. Moreover the hyperbolic diameter of K_n decreases to zero.*

Remark 1. *We can see that the closed set F in the above example satisfies the hypothesis of the Theorem, but does not satisfy the hypo-*

3) m is a one-dimensional Lebesgue measure.

thesis of C. Constantinescu and A. Cornea (Hilfssatz 19.3 *in* [3]).

Remark 2. *By Corollary* 1, *we see that a curve in a Stolz domain* $\Delta(\theta_0)$ *issuing from a point in U and terminating at* $z=1$ *is not thin at* $z=1$.

References

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