114. *Uniqueness of the Solution of Some Characteristic Cauchy Problems for First Order Systems*

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1. Introduction and definition. In this note we treat the following system;

$$A\frac{\partial u}{\partial t}=\sum_{j=1}^{n} B_j\frac{\partial u}{\partial x_j}+Cu, \tag{1.1}$$

where A, B_j $(j=1,\cdots,n)$ and C are all $N\times N$ matrices, and $u=u(t,x)$ is an N-vector. We consider the Cauchy problem for (1.1) with initial data on the hypersurface $t=0$. We concern here only with real analytic solutions, so we assume all the coefficients are real analytic in a neighborhood of $(t,x)=(0,0)$. If A is regular in a neighborhood of $(t,x)=(0,0)$, then we can find a unique solution for any analytic data by the well-known theorem of Cauchy-Kowalevskaya, however when A is singular at $t=0$ or in a neighborhood of $t=0$, many complicated affairs appear as for the existence or the uniqueness of the solution.

The case when A becomes singular only at $t=0$ was treated by M. Miyake [3], and he was concerned with the existence of the solution. For the single equation, one can refer to Y. Hasegawa [2].

The case which we treat here is that A is singular in a neighborhood of $(t,x)=(0,0)$, and we consider the uniqueness of the solution. In our case the uniqueness of the solution is deeply related to the lower order term, that is, to the matrix C.

To classify our equation, we give the following definition.

Definition 1.1. The equation (1.1) is said to be of type (p,q), if and only if the rank of A is p and the degree of $F(\tau;t,x)$ as a polynomial in τ is q in a neighborhood of $(t,x)=(0,0)$, where $F(\tau;t,x)$ denotes $det(\tau A-C)$.

Of course q does not exceed p, and if $p=N$, it is noncharacteristic case.

2. The case of constant coefficients. For the case of constant coefficients, we can obtain a necessary and sufficient condition for the solution of the Cauchy problem for (1.1) to be unique, if it is of type (p,p). Before stating the result, we refer to the following theorem which suggests the relation between the uniqueness of the solution and the lower order term.

Theorem 2.1. *Let us consider the following Cauchy problem;*

$$A\frac{du}{dt}=Bu+f(t), \qquad u(0)=\phi, \tag{2.1}$$

where A and B are $N\times N$ constant matrices, $f(t)$ and ϕ are given N-vectors and $u=u(t)$ is the unknown. Then a necessary and sufficient condition for the solution of (2.1) *to be unique is that $det(\tau A-B)\not\equiv 0$ as a polynomial in τ.*

According to Theorem 2.1, we can see that $det(\tau A-C)\not\equiv 0$ as a polynomial in τ is a necessary condition for the solution of the Cauchy problem for (1.1) to be unique. However, we can see easily it is not a sufficient one. For example, let $n=1$ and $A, B=B_1$ and C be as follows;

$$A=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \quad B=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad C=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix},$$

then $det(\tau A-C)=1-\tau$, but we have a null solution ${}^t(0, \omega(t)e^{-x})$, where $\omega(t)$ is an analytic function in t with $\omega(0)=0$. Thus, to obtain the uniqueness of the solution, we must impose some conditions on the first order terms in x.

To state our theorem, we prepare

Lemma 2.1. *Let the equation* (1.1) *be of type* (p, p) *and all the coefficients be constant, then the coefficient matrices A and C can be reduced to the following forms respectively;*

$$A\to\begin{pmatrix}E_p & 0\\ 0 & 0\end{pmatrix}, \quad \textit{and} \quad C\to\begin{pmatrix}S & 0\\ 0 & -E_{N-P}\end{pmatrix}, \tag{2.2}$$

where E_m denotes the $m\times m$ unit matrix and S a $p\times p$ matrix.

Now let us denote the first order operator in x in the reduced equation by

$$\begin{pmatrix}\alpha(D) & \mu(D)\\ \nu(D) & \beta(D)\end{pmatrix},$$

where D stands for $(\partial/\partial x_1\cdots, \partial/\partial x_n)$.

Our theorem is stated as

Theorem 2.2. *Let us consider the reduced equation for* (1.1). *Then a necessary and sufficient condition for the solution of the Cauchy problem to be unique is that the matrix $\beta(\xi)$ is nilpotent for any real unit vector $\xi=(\xi_1, \cdots, \xi_n)$.*

As for the existence of the solution, we have

Theorem 2.3. *Let $\beta(\xi)$ be nilpotent for any real unit vector ξ, then for an arbitrary polynomial Cauchy data $\phi(x)={}^t(P(x), Q(x))$ satisfying the following compatibility condition;*

$$\nu(D)P(x)+\beta(D)Q(x)-Q(x)=0, \tag{2.3}$$

there exists a solution of the Cauchy problem for the reduced equation.

3. The case of variable coefficients. In this section we are concerned with the case of variable coefficients. But it seems much difficult to obtain even a sufficient condition for the uniqueness of the solution for general equation. Therefore, we restrict ourselves to the equation of type $(N-1, N-1)$ and then we can assume $A=E_{N-1}$ in (1.1) without loss of generality. Moreover, we may assume the (N, N)-entry of the matrix C is -1, and let the (N, N)-entry of the matrix operator of the first order term in x be

$$\psi^1 \partial/\partial x_1 + \cdots + \psi^n \partial/\partial x_n. \tag{3.1}$$

Our result is stated as

Theorem 3.1. *Let* $\psi^j(0)=0$ $(j=1, \cdots, n)$. *If for any multi-index* $\alpha=(\alpha_1, \cdots, \alpha_n)$ *with components of non-negative integers, the following condition is satisfied*;

$$\alpha_1\lambda_1 + \cdots + \alpha_n\lambda_n \neq 1, \tag{3.2}$$

then the solution of the Cauchy problem for (1.1) *is unique, where* λ_j $(j=1, \cdots, n)$ *are the eigenvalues of the matrix* $(\partial\psi^j(0)/\partial x_k)$ $(j, k=1, \cdots, n)$.

The more detailed exposition will be published elsewhere.

References

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