112. A Note on the Abstract Cauchy Problem in a Banach Space

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§1. Introduction. This note is concerned with the abstract Cauchy problem for a linear operator A (with domain $D(A)$ and range $R(A)$) in a Banach space X. The problem considered here is to characterize the complete infinitesimal generator (or infinitesimal generator) of a semigroup of some class in terms of the abstract Cauchy problem. This problem was first treated by Hille and in [4], Phillips characterized the infinitesimal generator (simply i.g.) of a semigroup of class (C_0). His formulation of the abstract Cauchy problem (for a linear operator A) is as follows:

ACP—Given an element $x \in X$, fixed a function $u(t)=u(t;x)$ satisfying (i) $u(t)$ is strongly continuously differentiable in $t \geq 0$, (ii) $u(t) \in D(A)$ and $(d/dt)u(t)=Au(t)$ for each $t>0$ and (iii) $u(0;x)=x$.

A purpose of this note is to characterize the complete infinitesimal generator (c.i.g.) of a semigroup of class $(C_{(k)})$ in terms of ACP. But some properties of semigroups of class $(C_{(k)})$ $(k \geq 1)$ suggest the other formulation of the abstract Cauchy problem (see [3; p. 251]). For this sake, we introduced a less restrictive formulation:

WCP—Given an element $x \in X$, find a function $u(t)=u(t;x)$ satisfying (i′) $u(t)$ is strongly continuous in $t \geq 0$ and strongly continuously differentiable in $t>0$ and conditions (ii) and (iii) in ACP.

We shall call the X-valued function $u(t)$ satisfying (i) (or (i′)), (ii) and (iii) the solution of (APC; A, x) (or WCP; A, x)). In comparison with the solution of ACP, the behavior of the derivative of the solution of WCP has no restriction near $t=0$. Therefore, this formulation is called the weak Cauchy problem in [2] and is denoted by WCP in this note. However, the relationship between ACP and WCP when A has a nonvacuous resolvent set is described in Lemma 1.2.

Now, we state our result.

Theorem 1.1. *Let A be a closed linear operator with dense domain and nonvacuous resolvent set, and let k be a positive integer. Suppose that for each $x \in D(A^k)$ there is a unique solution $u(t;x)$ of* (WCP; A, x) *(or* (ACP; A, x)) *such that $u(t;x) \in D(A^k)$ for each $t>0$. Then A is the c.i.g. of a semigroup $\{T(t)\}_{t>0}$ of class $(C_{(k)})$ (or $(C_{(k-1)})$) such that $u(t;x)$*

$=T(t)x$ *for each* $x \in D(A^k)$.

Lemma 1.2. *Let A be a closed linear operator with nonvacuous resolvent set $\rho(A)$ and let $n \geq 1$ and $k \geq 1$ be integers. Suppose that $u(t)$ is a solution to* (WCP; A, x) *such that $u(t) \in D(A^k)$ then $v(t) = R(\lambda_0; A)^n u(t)$ is a solution to* (ACP; $A, R(\lambda_0; A)^n x$) *such that $v(t) \in D(A^{k+n})$ for all $t > 0$, where $\lambda_0 \in \rho(A)$ and $R(\lambda_0; A)$d enotes the resolvent of A.*

Lemma 1.2 gives two remarks on Theorem 1.1. First, we see that, if for every $x \in D(A^k)$ there is a unique solution $u(t; x)$ to (WCP; A, x) such that $u(t; x) \in D(A^k)$ then for every $y \in D(A^{k+1})$ there is a unique solution $v(t; y)$ to (ACP; A, y) such that $v(t; y) \in D(A^{k+1})$. Therefore, we may only consider the case when $u(t; x)$ of Theorem 1.1 is a solution of (ACP; A, x). Next, as for the uniqueness of the solution, we only assumed in Theorem 1.1 that, for every $x \in D(A^k)$ there is a unique solution $u(t;x)$ to (ACP; A, x) *such that* $u(t;x) \in D(A^k)$ for every $t > 0$. But this assumption and Lemma 1.2 imply that, for every $x \in D(A^k)$ there is a unique solution to (ACP; A, x).

Outline of the proof of Theorem 1.1 is given in § 3. Classes $(C_{(k)})$, $k=0, 1, \cdots$, of semigroups of bounded linear operators has recently been introduced by Oharu [3] and it is proved that the converse of Theorem 1.1 is true. Therefore, the c.i.g. of a semigroup of class $(C_{(k)})$ $(k \geq 1)$ is characterized in terms of both ACP and WCP. In § 2 of this note, we give a summary of basic properties of these semigroups and the converse of Theorem 1.1 is shown there. It is shown in [3; p. 255] that the class $(C_{(0)})$ is just the same as the familiar class (C_0). By virtue of this fact and Theorem 1.1, we obtain the first theorem in [4]: Let A be a closed linear operator with dense domain and nonvacuous resolvent set. Suppose that for each $x \in D(A)$ there is a unique solution $u(t; x)$ of (ACP; A, x). Then A is the i.g. of a semigroup $\{T(t)\}_{t>0}$ of class (C_0) such that $u(t; x) = T(t)x$ for all $x \in D(A)$.

In [4], Phillips also introduced another formulation of the abstract Cauchy problem, by imposing the following (1″) instead of (i) of ACP. (i″) $u(t)$ is strongly continuously differentiable in $t > 0$ and $\int_0^1 \|u'(t)\| dt < \infty$. This formulation is denoted by ACP_2 in [4]. The condition $\int_0^1 \|u'(t)\| dt < \infty$ is suggested by the property $\int_0^1 \|T(t)x\| dt < \infty$ of the semigroup $\{T(t)\}_{t>0}$ of class $(0, A)$. On the other hand, semigroups of class $(C_{(k)})$ $(k \geq 1)$ do not generally have this property. Therefore, we see that ACP_2 is not adequate to characterize the c.i.g. of a semigroup of class $(C_{(k)})$. The second theorem in [4] gives a characterization of the c.i.g. of a semigroup of class $(0, A)$ in terms of ACP_2. In view of the fact $(0, A) \subset (C_{(1)})$, this theorem may also be obtained through Theorem 1.1.

§ 2. Classes of semigroups. A family of bounded linear operators $\{T(t)\}_{t>0}$ on X to itself is called a *semigroup* if $T(t+s)=T(t)T(s)$ for $t, s>0$ and $T(t)$ is continuous in the strong operator topology for $t>0$. In this case, the *type* $\omega_0=\lim_{t\to\infty} t^{-1}\log\|T(t)\|<\infty$ is defined and the set $D(A_0)=\{x\in X\,;\, A_0x=\lim_{h\downarrow 0} h^{-1}(T(h)x-x)$ exists$\}$ is dense in $X_0=\bigcup_{t>0} T(t)(X)$. A_0 is called the *infinitesimal generator* of $\{T(t)\}_{t>0}$. We define $\sum=\{x\in X\,;\,\lim_{h\downarrow 0} T(h)x=x\}$ and call this the *continuity set*. Now, we consider a semigroup $\{T(t)\}_{t>0}$ with the properties:

(I) X_0 is dense in X,

(II) there is an $\omega_1>\omega_0$ such that for λ with $\lambda>\omega_1$ there is a bounded linear operator $R(\lambda)$ such that

$$R(\lambda)x=\int_0^\infty e^{-\lambda t}T(t)x\,dt \qquad \text{for } x\in X_0,$$

(III) if $R(\lambda)x=0$ for $\lambda>\omega_1$, then $x=0$.

For a semigroup satisfying (I)—(III), the infinitesimal generator A_0 is closable; the closure $\overline{A}_0=A$ is called the *complete infinitesimal generator*. Moreover A has the resolvent set $\rho(A)$ containing $\{\mathrm{Re}\,\lambda>\omega_1\}$ and

$$R(\lambda)=R(\lambda\,;A) \qquad \text{for } \lambda>\omega_1 \tag{2.1}$$

where $R(\lambda\,;A)$ denotes the resolvent of A.

Definition 2.1. Let $\{T(t)\}_{t>0}$ be a semigroup satisfying (I)—(III) and A be its c.i.g. Then $\{T(t)\}_{t>0}$ is said to be of class $(C_{(k)})$ if there is an integer $k\geq 0$ such that $D(A^k)\subset\sum$.

Class $(C_{(0)})$ is just the same as the familiar class (C_0). For a semigroup $\{T(t)\}_{t>0}$ of class $(C_{(k)})$, the following assertions hold:

(a) For every integer $l>0$, $(d/dt)^l T(t)x=A^l T(t)x=T(t)A^l x$ for $x\in D(A^l)$ and $t>0$,

(b) $T(t)x-x=\lim_{\delta\downarrow 0}\int_\delta^t AT(s)\,x\,ds$ for $x\in D(A^k)$,

(c) $T(t)x-x=\int_0^t AT(s)\,x\,ds$ for $x\in D(A^{k+1})$.

Proof and detailed explanations will be seen in Oharu [3; § 6]. The above assertions imply that for every $x\in D(A^k)$ (or $x\in D(A^{k+1})$) there is a solution $u(t\,;x)=T(t)x$ to (WCP; A, x) (or (ACP; A, x)) such that $u(t\,;x)\in D(A^k)$ (or $u(t\,;x)\in D(A^{k+1})$) for every $t>0$ and the uniqueness of the solution of ACP is proved in [3; p. 252]. This means the converse of Theorem 1.1.

§ 3. Outline of the proof of Theorem 1.1. Let $u(t\,;x)(\in D(A^k))$ be the solution to (ACP; A, x) ($x\in D(A^k)$). Define linear operators $\{U(t)\}_{t>0}$, on $D(A^k)$ to $D(A^k)$, by $x\mapsto U(t)x=u(t\,;x)$. In view of the uniqueness of the solution of ACP, we see that $U(t+s)=U(t)U(s)$ for $t, s>0$. By virtue of Lemma 1.2 and the ensuing remarks, we obtain, in the same way as in [4], the following

Lemma 3.1. (1) *For every* $T>0$ *there is an* $M_T>0$ *such that* $\|U(t)x\|_1 \leq M_T\|x\|_k$ *for* $0\leq t\leq T$ *and* $x\in D(A^k)$, *where* $\|x\|_k=\|x\|+\|Ax\|+\cdots+\|A^k x\|$. (2) *For every* $x\in D(A^{k+l})$, $l=1,2,\cdots$, *we have* $(d/dt)^l U(t)x=U(t)A^l x=A^l U(t)x$.

Henceforth we shall regard $D(A^k)$ as a Banach space $[D(A^k)]$ with respect to the norm $\|\cdot\|_k$. For every $t>0$, we apply the closed graph theorem to the operator $U(t)$ in $[D(A^k)]$ and we get

Lemma 3.2. *For every* $t>0$ *there is an* $M_t>0$ *such that* $\|U(t)x\|_k \leq M_t\|x\|_k$ *for* $x\in D(A^k)$.

By Lemmas 3.1 and 3.2, we see that the family of operators $\{U(t)\}_{t>0}$ has a unique extension to a semigroup $\{T(t)\}_{t>0}$ on X and the type ω_0 of $\{T(t)\}_{t>0}$ is defined as in § 2. Furthermore, for every $\omega>\omega_0$ there is an $M>0$ such that $\|T(t)x\|\leq Me^{t\omega}\|x\|_{k-1}$ for $t\geq 0$ and $x\in D(A^{k-1})$. Using these results and employing the same argument as in [3; p. 229], we get

Lemma 3.3. *The half plane* $\{\mathrm{Re}\,\lambda>\omega_0\}$ *is contained in the resolvent set* $\rho(A)$ *of* A *and we have*

$$R(\lambda;A)x=\int_0^\infty e^{-\lambda t}T(t)\,x\,dt \tag{3.1}$$

for $x\in D(A^{k-1})$ *and* $\mathrm{Re}\,\lambda>\omega_0$.

Proof of Theorem 1.1. First, we observe that $D(A^{k-1})\subset\sum$. Since $\{T(t)x\,;\,x\in D(A^{k-1}),\ t>0\}$ is dense in X and is contained in X_0, we get condition (I). From (3.1), we obtain conditions (II) and (III) and by (2.1), we see that A is the c.i.g. of the semigroup $\{T(t)\}_{t>0}$ of class $(C_{(k-1)})$. Therefore, the proof is complete.

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