107. *Limits of the Discrete Series for the Lorentz Groups*

By Toshiaki YONEYAMA
Osaka University

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1. Introduction. The purpose of this paper is to construct limits of the discrete series for the Lorentz group of n-th order and to show that the limits are imbedded in the principal series.

Limits of the discrete series have been constructed by Bargmann [1] for $SL(2, \boldsymbol{R})$ and by Takahashi [5] for the De Sitter group. The results in this paper is a generalization of them. Knapp and Okamoto [3] have discussed the same problem for limits of the holomorphic discrete series for a simple Lie group whose associated symmetric space has an invariant complex structure.

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2. Preliminaries. We denote by $Spin(n,1)$ the universal covering group of the Lorentz group $SO_e(n,1)$. $Spin(n,1)$ has been realized as a group consisting of 2×2 matrices with coefficients in the Clifford algebra by Takahashi [6] as follows: We use the same definitions and notations as in [6]. Let G be the set of matrices $g=\begin{pmatrix} a & b \\ b' & a' \end{pmatrix}$ such that

(2.1) $$a, b \in T_{n-1}, b\bar{a}' \in V_{n-1} \quad \text{and} \quad |a|^2-|b|^2=1.$$

Then G is a group, and if $n\geq 3$ G is isomorphic with $Spin(n,1)$. If $n=2$, G is isomorphic with $SU(1,1)$.

The subgroup K of G consisting of matrices $\begin{pmatrix} k & 0 \\ 0 & k' \end{pmatrix}$ with $k \in T^0_{n-1}$ is isomorphic with $Spin(n)$ and is a maximal compact subgroup of G. We identify $k \in T^0_{n-1}$ with $\begin{pmatrix} k & 0 \\ 0 & k' \end{pmatrix} \in K$ in the sequel.

3. Principal series. Let $G=KAN$ be the Iwasawa decomposition of G, and M the centralizer of A in K. Then the subgroups A, N and M consist of matrices of the form

$$a_t=\begin{pmatrix} \operatorname{ch} t/2 & \operatorname{sh} t/2 \\ \operatorname{sh} t/2 & \operatorname{ch} t/2 \end{pmatrix} (t \in R), \begin{pmatrix} 1-z & z \\ -z & 1+z \end{pmatrix} (z \in E_{n-1})$$

and

$$\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} (m=m' \in T^0_{n-1}),$$

respectively. M is isomorphic with $Spin(n-1)$. Let U and X be the spaces of $x \in V_{n-1}$ such that $|x|=1$ and $|x|<1$, respectively, then G acts on U and X to the left in the following way: for $g=\begin{pmatrix} a & b \\ b' & a' \end{pmatrix} \in G$ and $x \in U$(or X),

(3.1) $$g \cdot x=(ax+b)(b'x+a')^{-1}.$$

Put $(K/M)^*=K/M-\{kM\,;\, k+k'=0, k \in K\}$ and $U^*=U-\{-1\}$, then the map $p: G \rightarrow U$ defined by $\begin{pmatrix} a & b \\ b' & a' \end{pmatrix} \rightarrow (a+b)(b'+a')^{-1}$ gives an isomorphism of $(K/M)^*$ onto U^*. We define a map $s: U^* \rightarrow K$ by

$$s(u)=\begin{pmatrix} (1+u)/|1+u| & 0 \\ 0 & (1+\bar{u})/|1+u| \end{pmatrix},$$

then $p(s(u))=u$ for all $u \in U^*$ and every $g \in G$, except for a set of lower dimension, has a unique decomposition $g=s(p(g))m(g)a_{t(g)}z$, with $m(g) \in M$, $a_{t(g)} \in A$ and $z \in N$. For all $g \in G$ and $u \in U^*$, we have $p(gs(u)) = g \cdot u$ and

(3.2) $$gs(u)=s(g \cdot u)m(g, u)a_{t(g,u)}z,$$

with $m(g, u) \in M$, $a_{t(g,u)} \in A$ and $z \in N$. Let $d\mu(u)$ be the normalized K-invariant measure on U. Then we have

(3.3) $$d\mu(g \cdot u)=e^{(n-1)t(g,u)}d\mu(u), \quad \text{for } g \in G \quad \text{and} \quad u \in U,$$

and $e^{t(g,u)}$ and $m(g, u)$ are multipliers.

Every irreducible unitary representation of M is parametrized with a sequence $\lambda=(\lambda_1, \cdots, \lambda_{m-1})$ of integers or half-integers (half odd integers) such that

$\lambda_1 \geq \cdots \geq \lambda_{m-2} \geq |\lambda_{m-1}|$ if $n=2m+1$, $\lambda_1 \geq \cdots \geq \lambda_{m-2} \geq \lambda_{m-1} \geq 0$ if $n=2m$.

Let $(\sigma^\lambda, V^\lambda)$ be the irreducible unitary representation of M corresponding to such a sequence $\lambda=(\lambda_1, \cdots, \lambda_{m-1})$ as above and let $\mathcal{H}(\lambda)$ be the Hilbert space of functions $f: U \rightarrow V^\lambda$ such that

(3.4) $$\|f\|^2=\int_U \|f\|^2_{V^\lambda} d\mu(u) < \infty$$

where $\| \quad \|_{V^\lambda}$ denotes the norm in V^λ. Defining for $g \in G$ an operator $U_g(\lambda, \nu)$ ($\nu \in \boldsymbol{C}$) on $\mathcal{H}(\lambda)$ by

(3.5) $$U_g(\lambda, \nu)f(u)=e^{-\nu t(g^{-1},u)}\sigma^\lambda(m(g^{-1}, u))^{-1}f(g^{-1} \cdot u) \qquad (f \in \mathcal{H}(\lambda)),$$

we obtain a strongly continuous representation of G. If $\mathcal{R}e(\nu) =(n-1)/2$, $U(\lambda, \nu)$ is unitary and belongs to the principal series for G. It is known that $U(\lambda, \nu)$ is irreducible for any λ and ν if $n=2m+1$, and that it is irreducible unless λ is a sequence of half-integers and $\mathcal{I}m(\nu) =0$ if $n=2m$.

4. Discrete series. G has the discrete series if and only if $n=2m$, which we assume henceforth. We have seen in § 2 that G acts on X to the left by (3.1). The map $p_d: G \rightarrow X$ defined by $\begin{pmatrix} a & b \\ b' & a' \end{pmatrix} \rightarrow ba'^{-1}$ extends to an isomorphism of G/K onto X, and the map $s_d: X \rightarrow G/K$ defined by $s_d(x)=\begin{pmatrix} \operatorname{ch} t/2 & x \operatorname{ch} t/2 \\ \bar{x} \operatorname{ch} t/2 & \operatorname{ch} t/2 \end{pmatrix}$ ($\operatorname{th} t/2=|x|, t \geq 0$) is a section of p_d, i.e.,

$p_d(s_d(x))=x$ for all $x\in X$. Then every $g\in G$ has a unique decomposition $g=s_d(p_d(g))k(g)$ with $k(g)\in K$. According to this decomposition, we have

$$gs_d(x)=s_d(g\cdot x)k(g,x) \quad \text{with} \quad k(g,x)\in K. \tag{4.1}$$

If we denote by $d\mu(x)$ the Euclidean measure on X, then $(1-|x|^2)^{-2m}d\mu(x)$ is G-invariant.

Let $(\sigma^\lambda, V^\lambda)$ be the irreducible unitary representation of K corresponding to a sequence $\lambda=(\lambda_1,\cdots,\lambda_m)$ of integers or half-integers such that $\lambda_1\geq\lambda_2\geq\cdots\geq|\lambda_m|$, and let ν be an integer or half-integer $\geq m$. Let $H_1(\lambda,\nu)$ be the Hilbert space of functions $f: X\to V^\lambda$ such that

$$\|f\|^2=c\int_X \|f(x)\|^2_{V^\lambda}(1-|x|^2)^{2\nu-2m}d\mu(x)<\infty, \tag{4.2}$$

where c is a positive constant. Then $H_1(\lambda,\nu)$ is not 0. We define a unitary representation $g\to T_g(\lambda,\nu)$ of G on $H_1(\lambda,\nu)$ by

$$T_g(\lambda,\nu)f(x)=e^{-\nu t}(g^{-1},x)\sigma^\lambda(k(g^{-1},x))^{-1}f(g^{-1}\cdot x) \qquad (f\in H_1(\lambda,\nu)), \tag{4.3}$$

where $g^{-1}=\begin{pmatrix} a & b \\ b' & a'\end{pmatrix}$ and $e^{t(g,x)}=|b'x+a'|^2$.

Let $H_0(\lambda,\nu)$ be the subspace of $H_1(\lambda,\nu)$ of C^∞-functions, and let Ω denote the Casimir operator of G. We can then consider the operator $T_\Omega(\lambda,\nu)$ on $H_0(\lambda,\nu)$. We may identify the Lie algebra of G with that of $SO_e(2m,1)$. Using the notation in [5], we have for $f\in H_0(\lambda,\nu)$

$$\begin{aligned} -T_\Omega(\lambda,\nu)f=\Big[&(1-|x|^2)\Big\{\frac{1-|x|^2}{4}\Delta+(m-\nu-1)D \\ &+\sum_{i,j} x_j\sigma^\lambda(X_{ij})\frac{\partial}{\partial x_i}\Big\}+\sum_{i,j,k} x_jx_k\sigma^\lambda(X_{ij})\sigma^\lambda(X_{ik}) \\ &-\sum_{i<j}\sigma^\lambda(X_{ij})^2+\nu(\nu+m-1)|x|^2-m\nu\Big]f, \end{aligned} \tag{4.4}$$

where $\Delta=\frac{\partial^2}{\partial x_1^2}+\cdots+\frac{\partial^2}{\partial x_{2m}^2}$, $D=x_1\frac{\partial}{\partial x_1}+\cdots+x_{2m}\frac{\partial}{\partial x_{2m}}$ and $X_{ii}=0$, $X_{ij}+X_{ji}=0$ $(i\neq j)$.

For defining $T_\Omega(\lambda,\nu)$, we have followed [6].

To construct the discrete series for G in an analogous way to [5], we have to investigate $T_\Omega(\lambda,\nu)$ in more detail. Here we consider in a particular case that the parameters $\lambda_1,\cdots,\lambda_m$ of σ^λ satisfy

$$\lambda_1=\lambda_2=\cdots=\lambda_{m-1}=\lambda_m \quad \text{or} \quad \lambda_1=\lambda_2=\cdots=\lambda_{m-1}=-\lambda_m. \tag{4.5}$$

This is the case that the restriction of σ^λ to M is irreducible. For such σ^λ as above, making use of the results of Gelfand and Cejtlin [2], we have the following

Lemma 4.1. *Let λ,ν be integers or half-integers such that $\lambda\geq\nu\geq m$ and $\lambda-\nu$ is integer, and let $\sigma^{\lambda+}$* (resp. $\sigma^{\lambda-}$) *be the irreducible unitary representation of K corresponding to $\lambda^+=(\lambda,\cdots,\lambda,\lambda)$* (resp. λ^- $=(\lambda,\cdots,\lambda,-\lambda)$). *Then we have for $f\in H_0(\lambda^\pm,\nu)$,*

$$
\begin{aligned}
-T_{\Omega}(\lambda^{\pm},\nu)f=\Big[(1-|x|^2)\Big\{&\frac{1-|x|^2}{4}\Delta+(m-\nu-1)D \\
&+\sum_{i,j}x_j\sigma^{\lambda\pm}(X_{ij})\frac{\partial}{\partial x_i}\Big\}+(\nu(\nu-m+1) \\
&-\lambda(\lambda+m-1))|x|^2-(m\lambda(\lambda+m-1)-m\nu))\Big]f,
\end{aligned}
\tag{4.6}
$$

respectively.

Theorem 4.2. *Notations being as in Lemma 4.1, the subspaces $H(\lambda^{\pm},\nu)$ of $H_0(\lambda^{\pm},\nu)$ consisting of f such that*

$$T_{\Omega}(\lambda^{\pm},\nu)f=-[(m-1)\lambda(\lambda+m-1)+\nu(\nu-2m+1)]f \tag{4.7}$$

are non-trivial, closed and invariant for $T_g(\lambda^{\pm},\nu)$ $(g\in G)$, respectively. The unitary representations of G given by restricting $T(\lambda^{\pm},\nu)$ to $H(\lambda^{\pm},\nu)$ are irreducible and belong to the discrete series for G, respectively. Moreover, put $c=\dfrac{(2\nu-2m+1)\Gamma(\lambda-\nu+2m-1)\Gamma(\lambda+\nu)}{\Gamma(m)^2\Gamma(\lambda-\nu+m)\Gamma(\lambda+\nu-m+1)}$ in (4.2), then for any $v\in V^{\lambda\pm}$, the hypergeometric function $f_v(x)=F(\nu-\lambda-m+1,\lambda+\nu;m;|x|^2)v$ belongs to $H(\lambda^{\pm},\nu)$ and $\|f_v\|^2=\|v\|^2_{V^{\lambda\pm}}$, respectively.

Remark. In case $n=4$, our construction gives all the discrete series representations of the universal covering group of the De Sitter group (cf. [5]).

5. Limits of the discrete series and imbedding in the principal series. Let λ be a positive half-integer. We also denote by λ the sequence $(\lambda,\cdots,\lambda,\lambda)$ and let $(\sigma^{\lambda},V^{\lambda})$ be the corresponding irreducible unitary representation of K. The restriction of σ^{λ} to M is irreducible. For C^{∞}-function $f:X\to V^{\lambda}$, we define $T_g^+(\lambda)$ $(g\in G)$ by

$$T_g^+(\lambda)f(x)=e^{-(m-\frac{1}{2})t(g^{-1},x)}\sigma^{\lambda}(k(g^{-1},x))^{-1}f(g^{-1}\cdot x). \tag{5.1}$$

As in the case of the dicrete series, we have

$$
\begin{aligned}
-T_{\Omega}^+(\lambda)f=\Big[(1-|x|^2)\Big\{&\frac{1-|x|^2}{4}\Delta-\frac{1}{2}D+\sum_{i,j}x_j\sigma^{\lambda}(X_{ij})\frac{\partial}{\partial x_i} \\
&+\Big(\frac{1}{2}\Big(m-\frac{1}{2}\Big)-\lambda(\lambda+m-1)\Big)|x|^2+m\lambda(\lambda+m-1) \\
&-m\Big(m-\frac{1}{2}\Big)\Big]f.
\end{aligned}
\tag{5.2}
$$

Let $H_0^+(\lambda)$ be the space of C^{∞}-functions $f:X\to V^{\lambda}$ such that

$$T_{\Omega}^+(\lambda)f=-\Big[(m-1)\lambda(\lambda+m-1)-\Big(m-\frac{1}{2}\Big)^2\Big]f, \tag{5.3}$$

(5.4) f has a continuous extension to the boundary U, and

$$\|f\|^2=\int_U\|f(u)\|^2_{V^{\lambda}}d\mu(u)<\infty. \tag{5.5}$$

We define $T_g^-(\lambda)$ $(g\in G)$ by

$$T_g^-(\lambda)f(x)=e^{-(m-\frac{1}{2})t(g^{-1},x)}\sigma^{\lambda}(k'(g^{-1},x))^{-1}f(g^{-1}\cdot x), \tag{5.6}$$

where we denote by $k'(g,x)$ instead of $(k(g,x))'$. $H_0^-(\lambda)$ is defined similarly to $H_0^+(\lambda)$. Then $H_0^{\pm}(\lambda)$ are stable under $T_g^{\pm}(\lambda)$ $(g \in G)$, respectively. Let $H^{\pm}(\lambda)$ be the completions of $H_0^{\pm}(\lambda)$, respectively.

Lemma 5.1. *$f_v(x)=F(1/2-\lambda, \lambda+m-1/2; m; |x|^2)v$ $(v \in V^{\lambda})$ is a solution of* (5.3) *and* $\|f_v\|=(\Gamma(m)\Gamma(\lambda+1/2)/\Gamma(\lambda+m-1/2))\|v\|_{V\lambda}$. *Consequently, $H^{\pm}(\lambda)$ are not* 0.

Lemma 5.2. *The maps $I^{\pm}(\lambda)$ of $H^{\pm}(\lambda)$ into $\mathcal{H}(\lambda)$ (considering the restriction of σ^{λ} to M) defined by*

$$I^{\pm}(\lambda)f(u)=f(u) \qquad (f \in H_0^{\pm}(\lambda)), \tag{5.7}$$

are linear isometries, and for all $g \in G$

$$I^{\pm}(\lambda)T_g^{\pm}(\lambda)=U_g(\lambda, m-1/2)I^{\pm}(\lambda), \tag{5.8}$$

respectively.

Lemma 5.2 implies that $T_g^{\pm}(\lambda)$ $(g \in G)$ are extended to strongly continuous unitary representations of G on $H^{\pm}(\lambda)$, respectively and which are unitarily equivalent with subrepresentations of $U(\lambda, m-1/2)$.

Theorem 5.3. *The representations $T^{\pm}(\lambda)$ of G on $H^{\pm}(\lambda)$ are irreducible and mutually disjoint. Consequently, $U(\lambda, m-1/2)$ of the principal series for G is reducible.*

Remark. For $f \in H_0^{\pm}(\lambda)$, we have

$$\int_U \|f(u)\|^2_{V^\lambda} d\mu(u) = \frac{\Gamma(m)}{\pi^m} \lim_{\varepsilon \downarrow 0} \varepsilon \int_X \|f(x)\|^2_{V^\lambda}(1-|x|^2)^{\varepsilon-1} d\mu(x),$$

and hence taking account of the construction of the discrete series, we see that the representations $T^{\pm}(\lambda)$ on $H^{\pm}(\lambda)$ are limits of the discrete series.

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