132. *The Nonlinear Abstract Cauchy-Kowalewski Theorem described in the Form of Ranked Spaces*

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Introduction. Many interpretation of Cauchy-Kowalewski theorem can be seen in the various works. Main ones of them (cf. [1] p. 561) are classified as follows; (1) the classical interpretations (cf. [2] p. 16), (2) the generalized interpretation by T. Yamanaka [3] p. 7 or by L.V. Ovsjannikov [4] p. 819 (an immediate extension of Gelfand-Silov's result [5] p. 124), (3) the one by F. Treves [6] p. 77, and (4) the one by L. Nirenberg [1] p. 561. (1) is the one by using majorant series. (2), (3) and (4) are the one for an evolution equation by using Banach spaces scale regarded as a generalized majorant series. We denote it B.S. scale for short. (2) and (4) are the one for the equation with non-analytic coefficients in t. Nonlinear equations are treated only in (3) and (4). Now, let us show the unified interpretation of (1)$\sim$(4) (i.e. a generalization of the method of majorant series) by using ranked space [7] p. 3. Because ranked space (i.e. a generalization of uniform space by using transcendental ranks) is a generalization of B.S. scale in [4] p. 819, which is suitable for the description of conditional convergence (cf. E.R. integral in [7] p. 25) and for the description of the convergence in the set of germs. The elimination of parameter (by the norm) appearing in B.S. scale is aimed (in § 1) in the construction of ranked spaces by which we generalize the Cauchy-Kowalewski theorem to the one including (1), (2), (3) and (4). In § 2 we briefly discuss the relation pertaining Ovsjannikov's Theorem between our ranked space and B.S. scale.

§ 1. Cauchy-Kowalewski solution. **1°.** Let $\vec{x}\equiv(x_1,\cdots,x_n)$. Let $B_\delta^{n+1}=\{(s,x_1,x_2,\cdots,x_n);|s|<\delta,|x_i|<+\infty,i=1,2,\cdots,n\}$, let $\mathcal{F}_\delta^{(c)}$ be a set of continuous functions $C(B_\delta^{n+1})$ and let $\mathcal{F}_\delta\subset\mathcal{F}_\delta^{(c)}$. If the choice of $\mathcal{F}_\delta$ holds a sort of unicity, the equivalent relation $f_1\equiv f_2$ in $\bigcup_{\delta>0}\mathcal{F}_\delta$ for $f_1\in\mathcal{F}_{\delta_1}$ and $f_2\in\mathcal{F}_{\delta_2}$ defined by $f_1=f_2$ in $B^{n+1}_{\min(\delta_1,\delta_2)}$ satisfies the three axioms of equivalence. The set $\mathcal{F}_\delta^A$ of the analytic functions in s on B_δ^{n+1} is an example of this $\mathcal{F}_\delta$. The set consisting of the equivalent class $[f]$ for $f\in\bigcup_{\delta>0}\mathcal{F}_\delta$ is denoted by $\mathcal{F}$. The element of $\mathcal{F}$ becomes a germ (in a sense). **2°.** Suppose that $(\alpha f_1)(s,\vec{x})\equiv\alpha f_1(s,\vec{x})\in\mathcal{F}_{\delta_1}$ (for any real number α) and $(f_1+f_2)(s,\vec{x})\equiv f_1(s,\vec{x})+f_2(s,\vec{x})\in\mathcal{F}_{\min(\delta_1,\delta_2)}$ hold for $f_1\in\mathcal{F}_{\delta_1}$ and $f_2\in\mathcal{F}_{\delta_2}$, where $\delta_1,\delta_2>0$. Let $[f_1],[f_2]\in\mathcal{F}$ and let $[f]$ be the subset of $\bigcup_{\delta>0}\mathcal{F}_\delta$ corresponding to $[f]\in\mathcal{F}$. Let $\alpha[f_1]\equiv\{\alpha f; f\in[f_1]\}$ and let $[f_1]+[f_2]\equiv\{f+g; f\in[f_1], g\in[f_2]\}$.

Lemma 1. *If $\mathcal{F}$ is constructed by the solutions of a linear equation holding unicity, $\alpha[f_1]\in\mathcal{F}$ and $[f_1]+[f_2]\in\mathcal{F}$ holds.*

3°. Let $U_\delta(f)$ be a pre-neighbourhood of $f\in\mathcal{F}_\delta$ in $\mathcal{F}_\delta$ satisfying the following conditions; for any $U_\delta(f)$ and for any $\delta'\in(0,\delta)$ there exists $U_{\delta'}(f)$ such that $U_{\delta'}(f)\supseteq\{g+0; g\in U_\delta(f), 0\in\mathcal{F}_{\delta'}\}$ holds. Let $\tilde{U}_\delta(f)$ be the set of the element $[h]\in\mathcal{F}$ satisfying $[h]\subset\bigcup_{g\in U_\delta(f)}[g]$, and let $U^l(f)\equiv\bigcap_{|\delta|\leqslant l}\tilde{U}_\delta(f)$. Let I be a totally ordered set of limit or isolated ordinal numbers smaller than an inaccessible number. Let $\mathfrak{V}_i^l\equiv\{U_i^l(f)\}$ be a set of preneighbourhoods (like $U^l(f)$) such that $(\mathcal{F},\{\mathfrak{V}_i^l; i\in I\})$ becomes a ranked space [7] p. 3. 4°. Let $\mathcal{F}^{(0)}\equiv\{f(0,\vec{x}); f\in\bigcup_{\delta>0}\mathcal{F}_\delta\}$. Since $f_1\cong f_2$ for $f_1,f_2\in\bigcup_{\delta>0}\mathcal{F}_\delta$ defined by $f_1(0,\vec{x})=f_2(0,\vec{x})$ satisfies three axioms of equivalence, $\bigcup_{\delta>0}\mathcal{F}_\delta$ (therefore $\mathcal{F}$) can be also classified by this equivalence, and this classification derives $\tilde{\mathcal{F}}$ which has a natural one to one correspondence to $\mathcal{F}^{(0)}$. Let $\tau^{(0)}$ be a topology defined on $\mathcal{F}^{(0)}$ which makes $(\mathcal{F}^{(0)},\tau^{(0)})$ a uniform spaces. $\tilde{\tau}$ on $\tilde{\mathcal{F}}$ derived from $\tau^{(0)}$ makes $(\tilde{\mathcal{F}},\tilde{\tau})$ a uniform space. Furthermore $\tilde{\tau}$ is transformed to τ on $\mathcal{F}$ naturally.

Lemma 2. *$(\mathcal{F},\tau)$ is a uniform space* (cf. [8] p. 7).

Let $F(t)$ be a linear or non-linear mapping (dependent on t) of $\mathcal{F}^{(0)}$ into $\mathcal{F}^{(0)}$. (i) For example, let $[[\partial_s(F(s)u(s,\vec{x})]_{\partial_s u\equiv F(s)u}]_{s=0}\equiv[[s\Delta\partial_s u(s,\vec{x})+\Delta u(s,\vec{x})]_{\partial_s\equiv F(s)u}]_{s=0}=[s\Delta\{s\Delta u(s,\vec{x})\}+\Delta u(s,\vec{x})]_{s=0}=\Delta u(0,\vec{x})$ for $F(t)=t\Delta_x$. Let us apply this operation to general $F(s)$, and let $[\tilde{\partial}_s F(s)]_0 u(s,\vec{x})\equiv[[\partial_s(F(s)u(s,\vec{x}))]_{\partial_s u\equiv F(s)u}]_{s=0}$,

$$[(\tilde{\partial}_s)^2F(s)]_0\, u(s,\vec{x})\equiv[[\partial_s[\partial_s(F(s)u(s,\vec{x}))]_{\partial_s u\equiv F(s)u}]_{\partial_s u\equiv F(s)u}]_{s=0}$$

etc. (cf. [2] p. 20) for $u(s,\vec{x})\in\bigcup_{\delta>0}\mathcal{F}_\delta$. Suppose that $[(\tilde{\partial}_s)^i F(s)]_0$ (i; positive integer) is a mapping of $\mathcal{F}^{(0)}$ into $\mathcal{F}^{(0)}$. Let $f_n(t,\vec{x};u)=u(\vec{x})+tF(0)u(\vec{x})+\sum_{j=2}^n(t^j/j!)[[(\tilde{\partial}_s)^{j-1}F(s)]_0 u(s,\vec{x})]_{u(0,\vec{x})=u(\vec{x})}$. (ii) Let $u(t,\vec{x})=\int_0^t F(s)u(s,\vec{x})ds+u(0,\vec{x})\cdots(1)$ be the equation derived from $\partial_t u=F(t)u\cdots(2)$, and let $F(t)\equiv F_1(t)+F_2(t)$. Let $u_0(t,\vec{x};u)$ be a given function of $\vec{x}$ and t dependent on $u(t,\vec{x})$ with some regularity and $\{u_n(t,\vec{x};u); n=0,1,2,\cdots\}$ be a sequence of functions satisfying $u_n(t,\vec{x};u)=\int_0^t\{F_1(s)u_n(s,\vec{x};u)+F_2(s)u_{n-1}(s,\vec{x};u)\}ds+u(\vec{x})$. (iii) Hereafter let us use a suitable $\mathcal{F}_\delta$ ($\delta>0$) and $(\mathcal{F},\{\mathfrak{V}_n^l; n=1,2,\cdots\})$. Let $\mathfrak{V}_l\equiv[M; M\subset\tilde{\mathcal{F}}, \exists$Cauchy sequence $\{U_{n(i)}\}$ such that $\mathfrak{V}_{n(i)}^l\ni U_{n(i)}\ni f_{m(i)}(t,\vec{x};u)$ for $\forall u(\vec{x})\in M]$ and $\mathfrak{W}_l\equiv[M; M\subset\tilde{\mathcal{F}}, \exists$Cauchy sequence $\{U_{n(i)}\}$ such that $\mathfrak{V}_{n(i)}^l\ni U_{n(i)}\ni u_{m(i)}(t,x;u)$ for $\forall u(\vec{x})\in M]$, where $\{m(i)\}$ is a subsequence of $\{n\}$.

Example 1. $\{f_n(t,\vec{x};u); u\in\mathcal{F}^{(0)}, n=1,2,\cdots\}$, $\{u_n(t,\vec{x};u); u\in\mathcal{F}^{(0)},$

$n=1, 2, \cdots\}$, or linear hull of each one of them also becomes an example of $\mathfrak{F}$.

Theorem 1. (i) *$\mathfrak{B}_t \supseteq \mathfrak{B}_{t'}$ (or $\mathfrak{W}_t \supseteq \mathfrak{W}_{t'}$) holds for $0 \leqslant t \leqslant t'$.* (ii) *$(\tilde{\mathfrak{F}}, \{\mathfrak{B}_t ; 0 \leqslant t\})$ (or $(\tilde{\mathfrak{F}}, \{\mathfrak{W}_t ; 0 \leqslant t\})$) becomes a ranked space.*

Proof. (i) It follows from $U_{\delta'}(f) \supseteq \{g+0 ; g \in U_\delta(f), 0 \in \mathfrak{F}_{\delta'}\}$ $(0<\delta'<\delta)$ that $\mathfrak{B}_t \supseteq \mathfrak{B}_{t'}$ (or $\mathfrak{W}_t \supseteq \mathfrak{W}_{t'}$) holds for $0 \leqslant t \leqslant t'$. (ii) Since $\mathfrak{B}_l \ni M \supseteq \{u ; u \in M \subset \tilde{\mathfrak{F}}, \exists$Cauchy sequence $\{\tilde{U}_{n(i)}\}$ such that $\mathfrak{B}^{\tilde{t}}_{n(i)} \ni \tilde{U}_{n(i)}$ (for a given $\tilde{t} \geqslant l$) and $f_{m(i)}(t, \vec{x} ; u) \in \tilde{U}_{n(i)}\} \in \mathfrak{B}_{\tilde{t}}$ holds, $(\tilde{\mathfrak{F}}, \{\mathfrak{B}_t ; 0 \leqslant t\})$ becomes a ranked space. By the same way $(\tilde{\mathfrak{F}}, \{\mathfrak{W}_t ; 0 \leqslant t\})$ also becomes a ranked space.

Since the convergent $\{f_n(t, \vec{x} ; v)\}$ or $\{u_n(t, \vec{x} ; v)\}$ in the above (i), (ii) represent the solution of the equation (1) satisfying $u(0, x)=v(x)$, the following definition can be given.

Definition I. If a Cauchy sequence $\{U_t ; 0 \leqslant t \leqslant t_0\}$ in $(\tilde{\mathfrak{F}}, \{\mathfrak{B}_t ; 0 \leqslant t\})$ (or in $(\tilde{\mathfrak{F}}, \{\mathfrak{W}_t ; 0 \leqslant t\})$) is determined by $U^t_{n(i,t)}$ in $(\mathfrak{F}, \{\mathfrak{B}^t_n\})$ satisfying $\bigcap_{0 \leqslant t \leqslant t_0} \bigcap_{i=0}^{\infty} U^t_{n(i,t)} = \{f\}$ by only one $f \in \mathfrak{F}$, $\bigcap_{0 \leqslant t \leqslant t_0} U_t$ or $\{f\}$ is called the Cauchy-Kowalewski solution of the equation (1) with respect to $(\mathfrak{F}, \{\mathfrak{B}^t_n\})$ in $[0, t_0]$.

Remark 1. Since the analyticity in t (in $F(t)$) is not required in $\mathfrak{W}_t$, the interpretations (2) and (4) can be represented by $\mathfrak{W}_t$. Namely $F_1(s)=0$ and $F_2(s)=F(s)$ in (2), and $F_1(s)=$the linear (tangential) part of $F(s)$ developed at u_{n-1} and $F_2(s)=F(s)-F_1(s)$ in (4) [1] p. 566. Here $u_0(t, \vec{x} ; u) \equiv u_0(\vec{x})$.

Remark 2. $u_n(t, \vec{x} ; v)$ in the interpretation (2) becomes $v(\vec{x}) + \int_0^t F(s)v(x)ds + \sum_{i=2}^n \int_0^t F(s_{i-1}) \int_0^{s_{i-1}} F(s_{i-2}) \int_0^{s_{i-2}} \cdots \int_0^{s_1} F(s)v(\vec{x}) ds\, ds_1 \cdots ds_{i-1}$ which is similar to the form of $f_n(t, x ; v)$. (4) as a nonlinear extension of (2) is also a sort of perturbation.

Remark 2. $\begin{cases} (\square\text{-}m^2)\, u(x) = -G\bar{\psi}(x)\psi(x) \cdots (3) \text{ (cf. [9] p. 71).} \\ (i\gamma^\nu \partial / \partial x_\nu + M)\psi(x) = G\psi(x) \cdot u(x) \end{cases}$

The solution of (3) for $m=M=0$; $\varphi(x) = \pm\sqrt{3}/(2G) \cdot S^{-3/4}$, $\psi(x) = \pm[(x_0\gamma^0 - \sum_{\nu=1}^3 x_\nu\gamma^\nu)S^{-5/4} + iS^{-3/4}] \cdot \sqrt{3}\, a/(2G)\sqrt{|A|}$ and $\varphi(x) = \mp\sqrt{3}/(2G) \cdot S^{-3/4}$, $\psi(x) = \pm\sqrt{3}/(2G) \cdot [(x_0\gamma^0 - \sum_{\nu=1}^3 x_\nu\gamma^\nu)S^{-5/4} - iS^{-3/4}] \cdot a/\sqrt{|A|}$ cannot be obtained by the perturbation because of the appearance of $1/G$. Here $S=x_0^2-x_1^2-x_2^2-x_3^2$, $a=$constant spinor and $A=a^+\gamma^0 a$. This solution cannot be obtained either by (4) applied to the deformed equation of (3), because of $u_0(t, \vec{x} ; u) \equiv u_0(\vec{x})$.

§2. Ovsjannikov's theorem. $(\tilde{\mathfrak{F}}, \{\mathfrak{B}_t ; 0 \leqslant t\})$ and $(\tilde{\mathfrak{F}} \cdot \{\mathfrak{W}_t ; 0 \leqslant t\})$ (or their equivalents in $\mathfrak{F}$) are the practical examples of ranked space with non-parametric pre-neighbourhoods. If we continue a sort of specialization of it, the following spaces are derived; ranked space (a) with the pre-neighbourhoods dependent on one parameter, (b) with the pre-neighbourhoods dependent on norm (cf. B.S. scale), and (c) equivalent to B.S. scale with the norm $\|f, \Omega\|_{\rho, k+\alpha} = \sum_{l=0}^{\infty} \rho^l / l! \max_{|\beta|=l, \beta_0=0} \|D^\beta f\|_{C_{k+\alpha}(\Omega)}$ [10] p. 1350 (p. 45) having a physical application. Ovsjannikov's theorem [4] p. 1025 (p. 819) with respect to the convergent domain of the solution of the equation with a singular operator $F(t)$ in B.S. scale can be naturally extended to the space (a), and the norm $\|f, \Omega\|_{\rho, k+\alpha}$ can be described in the ranked space $\check{F}_R[\{1, 1, E^1(\Omega), D^\alpha\}, \tilde{\Gamma}]$ [11] p. 676. Namely $\{f; \|f, \Omega\|_{\rho, k+\alpha} < \bar{\varepsilon}\} = \bigcup_{\{\sum_{l=0}^{\infty} \bar{\varepsilon}_l \rho^l / l! < \bar{\varepsilon}\}} \bigcap_{l=0}^{\infty} \bigcup_{\varepsilon_l = \bar{\varepsilon}_l} \check{U}_l^* (0; \{[1, 1, \{1(\Omega)\}, \{D^{\beta+\alpha}; |\beta| = l + k, |\beta_0| \leqslant k\}]\}, \tilde{\Gamma}, \varepsilon)$ holds.

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