129. A Note on Nonsaddle Attractors

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(Comm. by Kunihiko KODAIRA, M. J. A., Oct. 12, 1973)

1. Introduction. We consider a dynamical system whose phase space X is a locally compact and connected metric space. Let M be a compact invariant set of this dynamical system. The purpose of this note is to prove the following:

Theorem. *If M is a nonsaddle positive attractor and $X-M$ contains at least one minimal set, then M is positively asymptotically stable whenever $A^+(M)-M$ is connected where $A^+(M)$ denotes the region of attraction of M.*

Definition of the terminology such as nonsaddle set, attractor, etc. will be given below. First we introduce the following notation.

For an arbitrary point x of X, we denote by:

(1) $C^+(x)$, the *positive half orbit* from x,
(2) $C^-(x)$, the *negative half orbit* from x,
(3) $L^+(x)$, the *positive limit set* of x,
(4) $L^-(x)$, the *negative limit set* of x,
(5) $D^+(x)$, the *positive prolongation* of x,
(6) $D^-(x)$, the *negative prolongation* of x,
(7) $J^+(x)$, the *positive prolongational limit set* of x,
(8) $J^-(x)$, the *negative prolongational limit set* of x.

Definition 1. The set

$$A^+(M)=[x\,;\, x\in X,\ M\supset L^+(x)\neq\emptyset],$$

is called the *region of positive attraction* of M, and the set

$$A^-(M)=[x\,;\, x\in X, M\supset L^-(x)\neq\emptyset]$$

is called the *region of negative attraction* of M.

Definition 2. The set

$$a^+(M)=[x\,;\, x\in X, M\cap L^+(x)\neq\emptyset]$$

is called the *region of positive weak attraction* of M, and the set

$$a^-(M)=[x\,;\, x\in X, M\cap L^-(x)\neq\emptyset]$$

is called the *region of negative weak attraction* of M.

Definition 3. M is called a *positive* (*negative*) *attractor* if $A^+(M)$ ($A^-(M)$) is a neighbourhood of M.

Definition 4. M is called a *saddle set* if there exists a neighbourhood U of M such that every neighbourhood of M contains a point x with the property that:

$$C^+(x) \not\subset U \quad \text{and} \quad C^-(x) \not\subset U.$$

Otherwise M is called a *nonsaddle set.*

2. Preliminary theorems. The proof of the Theorem depends upon several theorems already known. We shall state them here without proof.

Theorem 1. *If M is a positive attractor, then:*

1) *$D^+(M)$ is a positively asymptotically stable compact invariant set,*

2) *$A^+(M)$ is an open neighbourhood of $D^+(M)$, and*

$$A^+(D^+(M)) = A^+(M),$$

3) $D^+(M) = a^-(M)$.

Theorem 2. *If M is a nonsaddle set, then:*

1) *$L^+(x) \cap M \neq \emptyset$, $x \notin M$ implies $M \supset J^+(x) \supset L^+(x)$,*

2) *$L^-(x) \cap M \neq \emptyset$, $x \notin M$ implies $M \supset J^-(x) \supset L^-(x)$.*

Theorem 3. *If M is a nonsaddle set isolated from minimal sets (i.e. there exists a neighbourhood U of M such that $U - M$ contains no minimal sets), then $A^+(M) - M$ and $A^-(M) - M$ are both open sets.*

Theorem 1 is due to Auslander, Bhatia and Seibert [1]. Theorems 2 and 3 were proved by the author [2].

3. Proof of the Theorem. As M is a positive attractor, $A^+(M)$ is a neighbourhood of M. As is obvious from the definition of $A^+(M)$, $A^+(M) - M$ contains no minimal sets. Therefore M is isolated from minimal sets. Hence, by Theorem 3, $A^-(M) - M$ is an opes set.

By Theorem 1, $A^+(M)$ is an open set and

$$A^+(M) \supset D^+(M) = a^-(M).$$

As M is a nonsaddle set, it is easily seen from Theorem 2 that $a^-(M) = A^-(M)$. Therefore $D^+(M) = A^-(M)$. Since $D^+(M)$ is compact by Theorem 1, $A^-(M)$ is a compact subset of $A^+(M)$. Therefore $A^+(M) - A^-(M)$ is an open set.

As $A^+(M)$ and $A^-(M)$ both contain M, $A^+(M) - A^-(M)$ is a subset of $A^+(M) - M$ and we obtain

$$A^+(M) - M = (A^+(M) - A^-(M)) \cup (A^-(M) - M)$$

where $A^+(M) - A^-(M)$ and $A^-(M) - M$ are both open as was shown above, and obviously

$$(A^+(M) - A^-(M)) \cap (A^-(M) - M) = \emptyset.$$

So the connectedness of $A^+(M) - M$ implies either $A^+(M) - A^-(M)$ or $A^-(M) - M$ should be empty.

Suppose that $A^+(M) - A^-(M) = \emptyset$, or, what is the same thing, that $A^+(M) = A^-(M)$. Since $A^+(M)$ is open and $A^-(M) = D^+(M)$ is compact, this means that $A^+(M)$ is both closed and open. As X is connected, this implies

$$A^+(M) = X.$$

Since $A^+(M) - M$ contains no minimal sets, it follows from the above relation that $X-M$ cannot contain any minimal sets contrary to the assumption of the Theorem.

Therefore $A^{+}(M)-A^{-}(M)$ must be nonempty.

Consequently $A^{-}(M)-M=\emptyset$ and hence

$$M=A^{-}(M)=D^{+}(M).$$

As $D^{+}(M)$ is positively asymptotically stable by Theorem 1, M is positively asymptotically stable. Thus we have completed the proof of the Theorem.

References

[1] J. Auslander, N. P. Bhatia, and P. Seibert: Attractors in dynamical systems. Bol. Soc. Mat. Mexicana, **9**, 55–66 (1964).

[2] T. Saito: On a compact invariant set isolated from minimal sets. Funkcial. Ekvac., **12**, 193–203 (1969).