127. On Fourier Coefficients of Certain Cusp Forms

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(Comm. by Kunihiko KODAIRA, M. J. A., Oct. 12, 1973)

§ 1. Congruences on certain bases of $S_k(\Gamma)$.

We shall denote by Γ the group $SL(2, \boldsymbol{Z})$. The set of integral automorphic forms (cusp forms) of weight k (k being a positive integer) with respect to Γ forms a vector space $G_k(\Gamma)$ ($S_k(\Gamma)$) over the complex number field $\boldsymbol{C}$, whose dimension is known to be (cf. [7] p. 48):

$$\dim G_k(\Gamma)=\begin{cases}[k/12] & (k\equiv 2 \pmod{12}),\\ [k/12]+1 & (k\not\equiv 2 \pmod{12}),\end{cases}$$

$$\dim S_k(\Gamma)=\begin{cases}0 & (k=2),\\ [k/12] & (k\not\equiv 2 \pmod{12}),\\ [k/12]-1 & (k>2,\ k\equiv 2 \pmod{12}).\end{cases}$$

Any element $\varphi(\tau)$ ($\tau \in \boldsymbol{C}$, $\operatorname{Im} \tau>0$) of $G_k(\Gamma)$ admits a Fourier expansion in $q=e^{2\pi i\tau}$:

$$\varphi(\tau)=\sum_{n=0}^{\infty} \alpha(n)q^n;$$

we have $\varphi(\tau) \in S_k(\Gamma)$ if and only if $\alpha(0)=0$.

Using Eisenstein series, one can obtain bases of $S_k(\Gamma)$ as follows. Following the notation in [9], we put

$$\begin{aligned}E_k(\tau)&=\frac{1}{2}\sum_{\substack{l,m\in \boldsymbol{Z}\\ (l,m)\neq(0,0)}} (l\tau+m)^{-k} \qquad (k=4,6,8,\cdots)\\ &=\zeta(k)+\frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n\\ &=-\frac{(2\pi i)^k B_k}{2\cdot k!}+\frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n\end{aligned}$$

where $\zeta(s)$ is the Riemann zeta-function, B_k is the k-th Bernoulli number and

$$\sigma_g(n)=\sum_{\substack{t\mid n\\ t>0}} t^g \qquad (g=0,1,2,3,\cdots).$$

We put further

$$E_k^*(\tau)=1-\frac{2\cdot k}{B_k}\sum \sigma_{k-1}(n)q^n \qquad (k=4,6,8,\cdots), \tag{1}$$

so that

$$E_k(\tau)=\zeta(k)\cdot E_k^*(\tau).$$

Then the well-known cusp form $\Delta(\tau)$ of weight 12 under the name of Ramanujan's function is written in the form:

$$\Delta(\tau)=2^{-6}\cdot 3^{-3}(E_4^{*3}(\tau)-E_6^{*2}(\tau)).$$

$\Delta(\tau)$ has the Fourier expansion:

(2) $$\Delta(\tau)=\sum_{n=1}^{\infty} c(n)q^n, \quad \text{with} \quad c(n)\in \mathbf{Z}.$$

We have furthermore (cf. [7] p. 49)

$$S_k(\Gamma)=0 \qquad (k=4,6,8,10,14),$$
$$S_{12}(\Gamma)=[\Delta(\tau)],$$
$$S_k(\Gamma)=[\Delta(\tau)\cdot E_{k-12}^*(\tau)] \qquad (k=16,18,20,22),$$
$$S_{24}(\Gamma)=[\Delta(\tau)\cdot E_{12}^*(\tau)+\Delta^2(\tau)],$$
$$S_k(\Gamma)=[\Delta(\tau)\cdot E_4^{*a}(\tau)\cdot E_6^{*b}(\tau)]_{\substack{4a+6b=k-12\\ a,b\in \mathbf{Z},a,b\geqq 0}} \qquad (k\text{: even integer}>24)$$

i.e., $S_k(\Gamma)$ for even integer $k>24$ is spanned by the cusp forms of the type $\Delta(\tau)\cdot E_4^{*a}(\tau)\cdot E_6^{*b}(\tau)$, where a, b run over all non-negative integral solution of $4a+6b=k-12$.

We quote the following result of Siegel [9]. Let $k>2$ and $r=r_k=\dim G_k(\Gamma)$. Then the function $E_{12r-k+2}^*$ has the Fourier expansion of the form:

$$c_{k,r}q^{-r}+\cdots+c_{k,1}q^{-1}+c_{k,0}+\cdots,$$

where $c_{k,r}=1$ and

(3) $$\frac{c_{k,0}B_k}{2\cdot k}=\sum_{l=1}^{r} c_{k,l}\sigma_{k-1}(l).$$

Siegel has proved that $c_{k,0}\neq 0$. We put

(4) $$c_{k,0}=c_k.$$

Now we have the following propositions.

Proposition 1. *Let k be 16, 18, 20, or 22. Then the space $S_k(\Gamma)$ is spanned by the function $\Delta\cdot E_{k-12}^*$, which we denote by $f^{(k)}$. This function has the Fourier expansion of the form $f^{(k)}(\tau)=\sum_{n=1}^{\infty} a^{(k)}(n)q^n$. Then for the coefficients $a^{(k)}(n)$, we have*

(5) $$a^{(k)}(n)\equiv c(n) \qquad (\text{mod } c_{k-12})$$

for $n=1,2,3,\cdots$, where $c(n)$ are the coefficients in (2).

Remark 1. The values of c_{k-12} for $k=16,18,20,22$ are as follows (cf. [9]).

k	16	18	20	22
c_{k-12}	$-2^4\cdot 3\cdot 5$	$2^3\cdot 3^2\cdot 7$	$-2^5\cdot 3\cdot 5$	$2^3\cdot 3\cdot 11$

Remark 2. Our proposition may be more simply expressed by the following "congruence":

(5′) $$f^{(k)}(\tau)\equiv \Delta(\tau) \qquad (\text{mod } c_{k-12}).$$

Proof. Since

$$\Delta(\tau)=\sum_{n=1}^{\infty} c(n)q^n, \qquad E_{k-12}^*(\tau)=1-\frac{2(k-12)}{B_{k-12}}\sum \sigma_{k-13}(n)q^n,$$

so that

$$f^{(k)}(\tau)=\varDelta(\tau)\cdot E^*_{k-12}(\tau)=\sum_{n=1}^{\infty}c(n)q^n-\frac{2(k-12)}{B_{k-12}}\sum_{n=2}^{\infty}\sum_{i=1}^{n-1}c(i)\cdot\sigma_{k-13}(n-i)q^n.$$

As we have $r=1$ in (3), we have just $c_{k-12}=\dfrac{2(k-12)}{B_{k-12}}$, and as $c(i)\cdot\sigma_{k-13}(n-i)\in \boldsymbol{Z}$, we have the congruence (5) (cf. [8]).

Proposition 2. *Let k be any even integer >24 and f be a function of the form $\varDelta\cdot E_4^{*a}E_6^{*b}$ with $4a+6b=k-12$, i.e., any element of the base of the space $S_k(\Gamma)$ given above. Let the Fourier expansion of f be $f(\tau)=\sum_{n=1}^{\infty}b(n)q^n$. Then we have*

$$b(n)\equiv c(n)\pmod{24}$$

or more simply

$$f(\tau)\equiv\varDelta(\tau)\pmod{24}.$$

Proof. First we note the following fact. Let $\varphi(\tau)\in G_k(\Gamma)$ and

$$\varphi(\tau)=\varDelta(\tau)\cdot(1+m_1g_1(\tau))^a\cdot(1+m_2g_2(\tau))^b,$$

where a, b are non-negative integers, $m_1, m_2\in\boldsymbol{Z}$, and $g_1(\tau), g_2(\tau)$ admit Fourier expansions with integral coefficients in q. Then we have

$$\varphi(\tau)=\sum_{n=1}^{\infty}c(n)q^n+\sum_{n=1}^{\infty}d(n)q^n,$$

where $d(n)$ are integers divisible by the G.C.M. (m_1, m_2) of m_1 and m_2. This can be seen in the same way as in the proof of Proposition 1, in noticing that

$$\sum_{n=1}^{\infty}d(n)q^n=\varphi(\tau)-\sum_{n=1}^{\infty}c(n)q^n$$

can be written in the form:

$$m_1h_1(\tau)+m_2h_2(\tau)=(m_1, m_2)\cdot h_3(\tau),$$

where $h_i(\tau)$, $i=1,2,3$ admit Fourier expansions with integral coefficients in q.

The function $f=\varDelta\cdot E_4^{*a}\cdot E_6^{*b}$ in proposition 2 can be considered in virtue of (1) as a function φ above mentioned, where we have

$$m_1=-\frac{8}{B_4}=240,\qquad m_2=-\frac{12}{B_6}=-504,$$

so that $(m_1, m_2)=24$.

These propositions permit us to transfer the congruences which are known to hold on $c(n)$ to $a^{(k)}(n)$ or $b(n)$. The following are some examples of these congruences.

Firstly, it is known that

$$c(np)\equiv 0\pmod{p}$$

for all n if $p=2,3,5$ or 7 (cf. [2]).

From Proposition 1 follows therefore

$$a^{(16)}(np)\equiv a^{(20)}(np)\equiv 0\pmod{p}$$

for all n if $p=2,3$ or 5, and

$$a^{(18)}(np)\equiv 0\pmod{p}$$

for all n if $p=2,3$ or 7.

Secondly, from Proposition 2, we have

$$b^{(26)}(p)\equiv 1+p \pmod 3$$

for $p\neq 3$, since

$$c(p)\equiv 1+p \pmod 3$$

for $p\neq 3$, (cf. [5]). ($b^{(26)}(n)$ means the Fourier coefficient of $f=\Delta\cdot E_4^{*2}E_6^*$, which is a base of $S_{26}(\Gamma)$.)

§2. Effect of certain Hecke operators on bases of $S_k(\Gamma)$.

Let m be any positive integer. The m-th Hecke operator $T_k(m)$ operates on $\varphi\in G_k(\Gamma)$ as follows and gives an endomorphism of $G_k(\Gamma)$ and $S_k(\Gamma)$ respectively (cf. [3], [5]):

$$\varphi\,|\,T_k(m)=m^{k-1}\sum_{\substack{ad=m\\ a \bmod d,\, d>0\\ a,b,d\in Z}}\varphi\,|\begin{pmatrix} a & b\\ c & d\end{pmatrix}=m^{k-1}\sum_{\substack{ad=m\\ b \bmod d,\, d>0\\ a,b,d\in Z}}\varphi\left(\frac{a\tau+b}{d}\right)d^{-k}.$$

If $\varphi=\sum_{n=1}^{\infty}\alpha(n)q^n\in G_k(\Gamma)$ is an eigenfunction of $T_k(n)$ with the eigenvalue $\gamma(n)$, it is easily seen that

$$\gamma(n)\cdot\alpha(1)=\alpha(n).$$

$\varphi\in G_k(\Gamma)$ is said to be *normalized*, if $\alpha(1)=1$. For a normalized integral automorphic form φ, we have thus $\gamma(n)=\alpha(n)$.

Now we have the following theorems:

Theorem 1. *Let k be 16, 18 or 20, and p be a prime number such that $p\equiv -1 \pmod{c_{k-12}}\cdot f^{(k)}$ will denote the function $\Delta\cdot E_{k-12}^*$ as above. Then we have*

$$f^{(k)}\,|\,T_k(p)\equiv 0 \pmod{c_{k-12}}.$$

Theorem 2. *If $k=22$ or k is an any even integer >24 and f is a function the form $\Delta\cdot E_4^{*a}\cdot E_6^{*b}$ with $4a+6b=k-12$ and if $p\equiv -1 \pmod{24}$, then we have*

$$f\,|\,T_k(p)\equiv 0 \pmod{24}.$$

Proof of Theorem 1. As f is an eigenfuction of $T_k(p)$ with the eigenvalue $a^{(k)}(p)$, we have by virtue of Proposition 1:

$$f^{(k)}\,|\,T_k(p)=a^{(k)}(p)f^{(k)}\equiv c(p)f^{(k)} \pmod{c_{k-12}}.$$

Now the following congruences are known to hold:

$$c(p)\equiv 1+p^{11} \pmod{2^5}\ (p\neq 2)\quad \text{(cf. [1])},$$
$$c(p)\equiv p^2+p^9 \pmod{3^3}\quad \text{(cf. [4])},$$
$$c(p)\equiv p+p^{10} \pmod{5^2}\quad \text{(cf. [1])},$$
$$c(p)\equiv p+p^4 \pmod 7\quad \text{(cf. [6])}.$$

Hence follows $c(p)\equiv 0 \pmod{c_{k-12}}$ by our assumption $p\equiv -1 \pmod{c_{k-12}}$, which implies our conclusion.

Proof of Theorem 2. Let the Fourier expansion of f be:

$$f(\tau)=\sum_{n=1}^{\infty}b(n)q^n.$$

Then we have

$$f(\tau)\mid T_k(p)=p^{k-1}f(p\tau)+\frac{1}{p}\sum_{l \bmod p} f\left(\frac{\tau+l}{p}\right)$$
$$=p^{k-1}\sum_{n=1}^{\infty} b(n)q^{np}+\sum_{n=1}^{\infty} b(np)q^{n}.$$

From Proposition 2 we obtain, in noticing that $p^{k-12}\equiv 1 \pmod{24}$ for $k\geqq 22$, and that Δ is a common eigenfunction of Hecke operators $T_{12}(n)$, $n=1,2,3,\cdots$, the congruence

$$\begin{aligned} f(\tau)\mid T_k(p) &\equiv p^{k-1}\sum_{n=1}^{\infty} c(n)q^{np}+\sum_{n=1}^{\infty} c(np)q^{n} \pmod{24} \\ &= p^{k-12}\cdot p^{11}\sum_{n=1}^{\infty} c(n)q^{np}+\sum_{n=1}^{\infty} c(np)q^{n} \\ &\equiv p^{11}\sum_{n=1}^{\infty} c(n)q^{np}+\sum_{n=1}^{\infty} c(np)q^{n} \pmod{24} \\ &= \Delta(\tau)\mid T_{12}(p). \end{aligned}$$

Thus

(7) $$f(\tau)\mid T_k(p)\equiv c(p)\cdot\Delta(\tau) \pmod{24}.$$

From the congruences (6) we get $c(p)\equiv 0 \pmod{24}$ for $p\equiv -1 \pmod{24}$. Thus we obtain from (7) our conclusion:

$$f\mid T_k(p)\equiv 0 \pmod{24}.$$

Remark. In the case $k=22$, we have just $E_{10}^{*}=E_4^{*}\cdot E_6^{*}$. Theorem 1 can also be proved in the same way as in Theorem 2 but the proof would then become less simple. Theorem 2 for the case $k=22$ can also be proved like Theorem 1, as we have dim $S_{22}(\Gamma)=1$.

Acknowledgement. The author wishes to express his hearty thanks to Prof. S. Iyanaga, Prof. Y. Ihara and Mr. M. Koike for their kind advice.

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