126. *Complex Structures on $S^1 \times S^5$*

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1. Let X be a compact complex manifold of dimension 3 of which the 1st Betti number is equal to 1 and the 2nd Betti number vanishes. X has at most two algebraically independent meromorphic functions. In this note we restrict ourselves to the case where there are exactly two algebraically independent meromorphic functions. Then X has an algebraic net of elliptic curves. Furthermore we assume that this net has no base points, in other words, there exists a surjective holomorphic mapping f onto a projective algebraic (non-singular) surface V whose general fibre is an (connected, non-singular) elliptic curve. Finally we assume that f is equi-dimensional (see Remark 1). This note is a preliminary report on some results on complex structures of X. Details will be published elsewhere.

2. Proposition 1. *Every fibre of f is a non-singular elliptic curve.*

Proposition 2. *V is either a projective plane or a surface of general type.*

Theorem 1. *There exists an unramified covering manifold W of X such that $W \cup \{one\ point\}$ is holomorphically isomorphic to a 3-dimensional affine variety with an algebraic C^* action.*

Denote by α the linear transformation of N-dimensional complex affine space C^N defined by

$$\alpha: (z_1, \cdots, z_N) \mapsto (\alpha_1 z_1, \cdots, \alpha_N z_N),$$

where $\alpha_1^{a_1} = \cdots = \alpha_N^{a_N} = \beta$ for suitable positive integers a_j $(j=1, \cdots, N)$ and $0 < |\beta| < 1$. Then the infinite cyclic group $\langle \alpha \rangle$ generated by α acts on $C^N - \{0\}$ freely and the quotient space $C^N - \{0\}/\langle \alpha \rangle$ is a compact complex manifold.

Using some results of C. Chevally and M. Rosenlicht (see [8]), we obtain

Corollary. *There exists a finite unramified covering manifold of X which is holomorphically isomorphic to a submanifold of $C^N - \{0\}/\langle \alpha \rangle$ for some N and α.*

Let X_t be a small deformation of X. Then we have a small deformation W_t of W corresponding to X_t. By a theorem of H. Rossi [10], we obtain

Theorem 2. *$W_t \cup \{one\ point\}$ has a complex structure and be-*

comes a Stein space.

3. If X is topologically homeomorphic to $S^1 \times S^5$, then we have, as corollaries to Theorem 1 and 2, the following two theorems.

Theorem 3. *$\tilde{X} \cup \{$one point$\}$ is holomorphically isomorphic to a 3-dimensional affine variety with an algebraic C^* action where $\tilde{X}$ denotes the universal covering manifold of X.*

Theorem 4. *$\tilde{X}_t \cup \{$one point$\}$ has a complex structure and becomes a Stein space where $\tilde{X}_t$ denotes the universal covering manifold of X_t.*

4. In proving Proposition 1 and the key fact that $\dim H^1(X, \mathcal{O}_X) = 1$, the following lemma due to K. Akao plays a principal role.

Lemma. *Assume that a 3-dimensional compact complex manifold X is a fibre space over a non-singular projective algebraic surface V with projection f satisfying following two conditions,*

(1) *there exists a finite set of points $\{v_j; j=1, \cdots, \rho\}$ on V such that $f^{-1}(v)$ is an (non-singular) elliptic curve for each point $v \in V - \{v_j\}$,*

(2) $\dim H^1(X, \mathcal{O}_X) > \dim H^1(V, \mathcal{O}_V)$.

Then there exists a following exact sequence of sheaves

$$0 \to \mathcal{O}_V \to R^1 f_* \mathcal{O}_X \to \mathcal{S} \to 0,$$

where the support of $\mathcal{S}$ is a finite set of points.

Remark 1. K. Akao proved that $f: X \to V$ is always equi-dimensional with the aid of the spectral sequence

$$E_2^{p,q} = H^p(V, R^q f_* C) \Rightarrow H^{p+q}(X, C).$$

Remark 2. Let S be an elliptic surface of which the 1st Betti number is odd (i.e., a VI_0- class surface or a VII_0- class elliptic surface (K. Kodaira [7])). Then every fibre of S is a non-singular elliptic curve. By similar methods of proofs, we can show that the statements corresponding to Theorem 1 and Corollary are valid.

Example (Brieskorn and Van de Ven [1]). Let $a = (a_0, a_1, a_2, a_3)$ be a 4 tuple of positive integers and (z_0, z_1, z_2, z_3) the standard coordinate in C^4. We define an affine variety $X(a) = X(a_0, a_1, a_2, a_3)$ by the equation

$$z_0^{a_0} + z_1^{a_1} + z_2^{a_2} + z_3^{a_3} = 0.$$

Let S^7 be the sphere $\sum_{i=0}^{3} |z_i|^2 = 1$. Then the space

$$\textstyle\sum (a) = X(a) \cap S^7$$

is a 5-dimensional oriented differentiable manifold in a natural way. Let $\langle \alpha \rangle$ be the infinite cyclic group acting on $X(a) - \{0\}$ freely generated by the transformation

$$\alpha: (z_0, z_1, z_2, z_3) \mapsto (\alpha^{a_0^{-1}} z_0, \alpha^{a_1^{-1}} z_1, \alpha^{a_2^{-1}} z_2, \alpha^{a_3^{-1}} z_3),$$

where $\alpha \in C^*$ and $|\alpha| \neq 1$. Then $H(a) = X(a) - \{0\} / \langle \alpha \rangle$ is a compact complex manifold of dimension 3. By Proposition 4 in [1], if a_i and a_j are mutually prime for any $i \neq j$, then $H(a)$ is an elliptic fibre space over a projective plane whose topological model is $S^1 \times S^5$. Every fibre of $H(a)$ is an (connected, non-singular) elliptic curve.

References

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