152. *Some Radii of a Solid Associated with Polyharmonic Equations*

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Introduction. In the preceding paper [1], we treated some quantities of a bounded domain in R^2 which we called polyharmonic inner radii. In the present paper, we deal with the similar quantities of a bounded domain in R^3 which is bounded by finite number of regular surfaces. G. Pólya and G. Szegö [2] defined the inner radius of a bounded domain using the Green's function of the domain relative to the Laplace's equation $\Delta u=0$ and they calculated the inner radius of a nearly spherical domain. The aim of this paper is to extend the above results. In the first place, we obtain the Green's function of a sphere relative to the n-harmonic equation $\Delta^n u=0$ and define the n-harmonic inner radius of a bounded domain. In the next place, we compute the n-harmonic inner radius of a nearly spherical domain and it is noticeable that it is monotonously decreasing with respect to integer n.

1. Inner radii associated with polyharmonic equations.

We use the following notations in this section. Let V be a bounded domain in R^3, S the surface of V, P_0 an inner point of V, P the variable point in V and r the distance from P_0 to P.

Definition 1. If a function $u(P)$ satisfies the following two conditions, $u(P)$ is called the Green's function of V with the pole P_0 relative to the n-harmonic equation $\Delta^n u=0$.

(1) In a neighborhood of P_0, $u(P)$ has the form

$$u(P)=r^{2n-3}+h_n(P),$$

where $h_n(P)$ satisfies the equation $\Delta^n h_n=0$ in V and all its derivatives of order $\leqq 2n-1$ are continuous in $V+S$.

(2) All the normal derivatives of order $\leqq n-1$ of $u(P)$ vanish on S.

We can find the Green's function relative to the equation $\Delta^n u=0$ for a sphere in the explicit form.

Theorem 1. *Let V be the sphere of radius R with the center O. If $P_0\neq O$, denoting ρ the distance from O to P_0, P_0' the inversion of P_0 with respect to S and r' the distance from P_0' to P, the Green's function $G_n(P,P_0)$ of V with the pole P_0 relative to the equation $\Delta^n u=0$ is as* *follows,*

$$G_1(P, P_0)=\frac{1}{r}-\frac{R}{\rho r'},$$

$$G_n(P, P_0)=-\frac{R}{2\rho r'}\left\{r^{2n-4}\left(r-\frac{\rho}{R}r'\right)^2 - \sum_{k=2}^{n-1}\frac{(2k-2)!}{2^{2k-2}k!(k-1)!}r^{2n-2k-2}\left(r^2-\frac{\rho^2}{R^2}r'^2\right)^k\right\} \quad (n\geqq 2).$$

And $P_0=O$, we put $\rho r'=R^2$ in the above equalities.

Proof. The case of $n=1$ is well known. Obviously the function $G_n(P, P_0)$ $(n\geqq 2)$ satisfies the condition (1). We write the function $G_n(P, P_0)$ in the form

$$G_n(P, P_0)=-\frac{Rr^{2n-2}}{2\rho r'}\left[\left(1-\frac{\rho r'}{Rr}\right)^2-\sum_{k=2}^{n-1}\frac{(2k-2)!}{2^{2k-2}k!(k-1)!}\left\{1-\left(\frac{\rho r'}{Rr}\right)^2\right\}^k\right],$$

and we put

$$x=\left(\frac{\rho r'}{Rr}\right)^2.$$

Then x is equal to 1 on S, and we can rewrite

$$G_n(P, P_0)=-\frac{Rr^{2n-2}}{2\rho r'}\left\{(1-\sqrt{x})^2-\sum_{k=2}^{n-1}\frac{(2k-2)!}{2^{2k-2}k!(k-1)!}(1-x)^k\right\}.$$

If we put

$$f_n(x)=(1-\sqrt{x})^2-\sum_{k=2}^{n-1}\frac{(2k-2)!}{2^{2k-2}k!(k-1)!}(1-x)^k,$$

then

$$f_n^{(\alpha)}(1)=0, \qquad 0\leqq\alpha\leqq n-1.$$

From this the condition (2) follows. The theorem is thus proved.

Given a domain V and an inner point P_0 of V, G. Pólya and G. Szegö [2] defined the inner radius r_{P_0} of V with respect to the point P_0 as follows; when the Green's function $G(P, P_0)$ of V with the pole P_0 relative to the equation $\Delta u=0$ is

$$G(P, P_0)=\frac{1}{r}+h(P),$$

they put

$$\frac{1}{r_{P_0}}=-h(P_0).$$

Now we define the n-harmonic inner radius of a domain V associated with n-harmonic equation $\Delta^n u=0$.

Definition 2. If the Green's function of a domain V with the pole P_0 relative to the equation $\Delta^n u=0$ is

$$r^{2n-3}+h_n(P),$$

then we put

$$\frac{1}{r_{P_0,1}}=-h_1(P_0),$$

$$\frac{(2n-4)!}{2^{2n-3}(n-1)!\,(n-2)!} r_{P_0,n}^{2n-3} = |h_n(P_0)| \qquad (n \geqq 2),$$

we call $r_{P_0,n}$ the n-harmonic inner radius of the domain V with respect to the point P_0.

Remark. When the domain V is the sphere of radius R, we compute the ordinary inner radius and the n-harmonic inner radius of the sphere with respect to the point P_0, which are the same value

$$\frac{R^2-\rho^2}{R}$$

for an arbitrary integer n.

2. Inner radii of a nearly spherical domain.

In this section, we consider the radii of a nearly spherical domain defined in the former section.

Definition 3. Let

$$(1) \qquad r = 1 + \rho(\theta, \varphi)$$

be the equation of the surface of a domain in spherical coordinates r, θ and φ, where $\rho(\theta, \varphi)$ represents the infinitesimal variation of the unit sphere. We call the domain bounded by (1) the nearly spherical domain.

We consider the series

$$(2) \qquad \rho(\theta, \varphi) = \sum_{k=0}^{\infty} X_k(\theta, \varphi),$$

where the term $X_k(\theta, \varphi)$ represents a surface harmonic of degree k which has infinitesimal coefficients of the first order.

Terms of higher infinitesimal than the second order are neglected in all the discussions of this section.

Lemma. *When the surface harmonic $X_1(\theta, \varphi)$ in the series (2) is*

$$X_1(\theta, \varphi) = (a \cos \varphi + b \sin \varphi) \sin \theta + c \cos \theta,$$

neglecting terms of higher than the first order, the position of the centroid C of the nearly spherical domain $r < 1 + \rho(\theta, \varphi)$ is (a, b, c).

This lemma was given by G. Pólya and G. Szegö [2] and they obtained the ordinary inner radius r_C of the nearly spherical domain with respect to the centroid C as follows,

$$(3) \qquad r_C = 1 + X_0 + \frac{1}{4\pi}\left\{\iint [X_1(\theta, \varphi)]^2 dS - \sum_{k=2}^{\infty} (k+1) \int [X_k(\theta, \varphi)]^2 dS\right\},$$

where the integral is extended over the surface of the unit sphere of which dS is an element. As an extention of (3), we prove the following theorem.

Theorem 2. *For an arbitrary positive integer n, the n-harmonic inner radius $r_{C,n}$ of the nearly spherical domain $r < 1 + \rho(\theta, \varphi)$ with respect to the centroid C is*

$$(4) \qquad r_{C,n} = 1 + X_0 + \frac{1}{4\pi}\left\{\iint [X_1(\theta, \varphi)]^2 dS - \sum_{k=2}^{\infty} (nk-n+2) \int [X_k(\theta, \varphi)]^2 dS\right\}.$$

Consequently, $r_{C,n}$ decreases monotonously with respect to n.

Proof. We seek the Green's function $G_n(P,C)$ of the nearly spherical domain with the pole C relative to the equation $\Delta^n u=0$ in the form

$$G_n(P,C)=-\frac{1}{2}\left\{r'^{2n-4}(r'-1)^2-\sum_{k=2}^{n-1}\frac{(2k-2)!}{2^{2k-2}k!\,(k-1)!}r'^{2n-2k-2}(r'^2-1)^k\right\}$$
$$+p(r,\theta,\varphi)+q(r,\theta,\varphi),$$
$$p(r,\theta,\varphi)=\sum_{k=0}^{\infty}\sum_{i=0}^{n-1}r^{k+2i}S_k^{(i)}(\theta,\varphi),$$
$$q(r,\theta,\varphi)=\sum_{k=0}^{\infty}\sum_{i=0}^{n-1}r^{k+2i}T_k^{(i)}(\theta,\varphi).$$

Here r' is the distance from $C(a,b,c)$ to the point P, $S_k^{(i)}(\theta,\varphi)$ and $T_k^{(i)}(\theta,\varphi)$ are surface harmonics of degree k with first and second order coefficients respectively. The n-harmonic inner radius $r_{C,n}$ is determined by

$$\frac{(2n-4)!}{2^{2n-3}(n-1)!\,(n-2)!}r_{C,n}^{2n-3}$$
$$=\left|(-1)^{n-1}\frac{(2n-4)!}{2^{2n-3}(n-1)!\,(n-2)!}+p(r_0,\theta_0,\varphi_0)+q(r_0,\theta_0,\varphi_0)\right|,$$

where r_0,θ_0 and φ_0 are spherical coordinates of the centroid C. And so we have

$$r_{C,n}^{2n-3}=1+(-1)^{n-1}\frac{2^{2n-3}(n-1)!\,(n-2)!}{(2n-4)!}\{S_0^{(0)}+(rS_1^{(0)})+T_0^{(0)}\},\tag{5}$$

where the term $(rS_1^{(0)})$ has to be taken at C. If ν denotes the normal to the boundary of the nearly spherical domain, the condition

$$\frac{\partial^m G_n}{\partial\nu^m}=0$$

on the boundary can be replaced by

$$\frac{\partial^m G_n}{\partial r^m}=0.$$

Let γ be the variable angle between the radii r and r_0. In view of lemma, we have

$$r_0\cos\gamma=X_1(\theta,\varphi).$$

Now the boundary conditions are

$$\frac{\partial^\alpha}{\partial r^\alpha}p(1,\theta,\varphi)+\rho(\theta,\varphi)\frac{\partial^{\alpha+1}}{\partial r^{\alpha+1}}p(1,\theta,\varphi)+\frac{\partial^\alpha}{\partial r^\alpha}q(1,\theta,\varphi)=0$$
$$0\leqq\alpha\leqq n-3,$$

$$\frac{\partial^{n-2}}{\partial r^{n-2}}p(1,\theta,\varphi)+\rho(\theta,\varphi)\frac{\partial^{n-1}}{\partial r^{n-1}}p(1,\theta,\varphi)+\frac{\partial^{n-2}}{\partial r^{n-2}}q(1,\theta,\varphi)$$
$$=\frac{(2n-2)!}{2^n(n-1)!}\{\rho(\theta,\varphi)-r_0\cos\gamma\}^2,\tag{6}$$
$$\frac{\partial^{n-1}}{\partial r^{n-1}}p(1,\theta,\varphi)+\rho(\theta,\varphi)\frac{\partial^{n}}{\partial r^{n}}p(1,\theta,\varphi)+\frac{\partial^{n-1}}{\partial r^{n-1}}q(1,\theta,\varphi)$$

$$=\frac{(2n-2)!}{2^{n+1}(n-1)!}[(n^2+n-6)\{\rho(\theta,\varphi)-r_0\cos\gamma\}^2+2r_0^2\sin\gamma]$$
$$+\frac{(2n-2)!}{2^{n-1}(n-1)!}\{\rho(\theta,\varphi)-r_0\cos\gamma\}.$$

The first order terms yield

$$\frac{\partial^\alpha}{\partial r^\alpha}p(1,\theta,\varphi)=0 \qquad 0\leqq\alpha\leqq n-2,$$
$$(7)\qquad \frac{\partial^{n-1}}{\partial r^{n-1}}p(1,\theta,\varphi)=\frac{(2n-2)!}{2^{n-1}(n-1)!}\{\rho(\theta,\varphi)-r_0\cos\gamma\}.$$

So that

$$(8)\qquad p(r,\theta,\varphi)=\frac{(2n-2)!}{2^{2n-2}\{(n-1)!\}^2}(r^2-1)^{n-1}\left\{X_0+\sum_{k=2}^{\infty}r^kX_k(\theta,\varphi)\right\};$$

in particular

$$(9)\qquad S_0^{(0)}=(-1)^{n-1}\frac{(2n-2)!}{2^{2n-2}\{(n-1)!\}^2}X_0,\qquad (rS_1^{(0)})=0.$$

We consider the second order terms. The mean value of the function $q(r,\theta,\varphi)$ on the surface of the unit sphere is equal to $\sum_{i=0}^{n-1}r^{2i}T_0^{(i)}$. By the first equations of (6) and (7) we have

$$\frac{\partial^\alpha}{\partial r^\alpha}q(1,\theta,\varphi)=0 \qquad 0\leqq\alpha\leqq n-3,$$

so that it must be the form

$$(10)\qquad \sum_{i=0}^{n-1}r^{2i}T_0^{(i)}=(r^2-1)^{n-2}(Ar^2+B),$$

where A and B are constants. Comparing the constant coefficients of $q(r,\theta,\varphi)$ and (10), we obtain

$$(11)\qquad T_0^{(0)}=(-1)^{n-2}B.$$

We take now the mean values of second order terms of (6) and find

$$A+B=-\frac{(2n-2)!}{2^{2n-2}(n-1)!\,(n-2)!}\left\{X_0^2+\sum_{k=2}^{\infty}\frac{1}{4\pi}\int[X_k(\theta,\varphi)]^2dS\right\},$$

$$(n+2)A+(n-2)B$$
$$=-\frac{(2n-2)!}{2^{2n-2}\{(n-1)!\}^2}\left[(n^2-3n+6)\left\{X_0^2+\sum_{k=2}^{\infty}\frac{1}{4\pi}\int[X_k(\theta,\varphi)]^2dS\right\}\right.$$
$$\left.+4n\sum_{k=2}^{\infty}\frac{k}{4\pi}\int[X_k(\theta,\varphi)]^2dS-\frac{4}{4\pi}\int[X_1(\theta,\varphi)]^2dS\right].$$

Consequently

$$(12)\qquad B=\frac{(2n-2)!}{2^{2n-2}\{(n-1)!\}^2}\left[(-n+2)\left\{X_0^2+\sum_{k=2}^{\infty}\frac{1}{4\pi}\int[X_k(\theta,\varphi)]^2dS\right\}\right.$$
$$\left.+n\sum_{k=2}^{\infty}\frac{k}{4\pi}\int[X_k(\theta,\varphi)]^2dS-\frac{1}{4\pi}\int[X_1(\theta,\varphi)]^2dS\right].$$

By virtue of (5), (9), (11) and (12) we find

$$r_{C,n}=1+X_0+\frac{1}{4\pi}\left\{\int[X_1(\theta,\varphi)]^2dS-\sum_{k=2}^{\infty}(nk-n+2)\int[X_k(\theta,\varphi)]^2dS\right\}.$$

This is the desired equation.

References

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