151. A Note on the Perturbing Uniform Asymptotically Stable Systems

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1. Introduction. Recently, A. Strauss and J. A. Yorke [2], [3] obtained results concerning the perturbation of the eventual uniform asymptotic stability (abbreviated by *EvUAS*). They considered

(E) $x'=f(t,x)$

(P–1) $y'=f(t,y)+g(t,y)$

(P–2) $y'=f(t,y)+h(t)$

(P–3) $y'=f(t,y)+g(t,y)+h(t)$

under the condition that $f(t,x)$ satisfies the Lipschitz condition and $g(t,x)$ and $h(t)$ satisfy various conditions. In this note we consider these problems under the generalized Lipschitz conditions.

We shall assume that x, f, g and h are n-vectors in R^n and $|\cdot|$ is some n-dimensional norm. Moreover we shall assume that $x=0$ is *EvUAS* for (E) and g and h are smooth for the local existence.

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2. Definitions and auxiliary lemma. In what follows, we denote by $x(t;t_0,x_0)$ any solution of (E) through the point (t_0,x_0).

Definition 2.1. The origin 0 of R^n is said to be for the system (E):

(E_1) *eventually uniformly stable* (*EvUS*) if, for every $\varepsilon>0$, there exist $\alpha=\alpha(\varepsilon)\geqq 0$ and $\delta=\delta(\varepsilon)>0$ such that

$$|x(t;t_0,x_0)|<\varepsilon \qquad \text{for } |x_0|<\delta \text{ and } t\geqq t_0\geqq\alpha;$$

(E_2) *eventually uniformly attracting* (*EvUA*) if, there exist $\delta_0>0$ and $\alpha_0\geqq 0$ and if for every $\varepsilon>0$ there exists $T=T(\varepsilon)\geqq 0$ such that

$$|x(t;t_0,x_0)|<\varepsilon \qquad \text{for } |x_0|<\delta_0,\ t_0\geqq\alpha_0 \text{ and } t\geqq t_0+T;$$

(E_3) *eventually uniform-asymptotically stable* (*EvUAS*) if (E_1) and (E_2) hold simultaneously.

Definition 2.2. A continuous function $h:[0,\infty)\to R^n$ is said to be *absolutely diminishing* if

$$\int_t^{t+1}|h(s)|ds\to 0 \qquad \text{as } t\to\infty.$$

A continuous function $g:[0,\infty)\times R^n\to R^n$ is said to be *absolutely diminishing* if for some $r>0$ and every $m(0<m<r)$, there exists an

absolutely diminshing function $h_m(t)$ such that $|g(t,x)|\leqq h_m(t)$ for all $t\geqq 0$ and $m\leqq|x|\leqq r$.

Definition 2.3. A continuous function $h:[0,\infty)\to R^n$ is said to be *diminishing* if

$$\sup_{0\leqq u\leqq 1}\left|\int_t^{t+u} h(s)ds\right|\to 0 \qquad \text{as } t\to\infty.$$

A continuous function $g:[0,\infty)\times R^n\to R^n$ is said to be *diminishing* if the following two conditions are satisfied;

(D–1) for some $r>0$ and each fixed x $(0<|x|\leqq r)$

$$\sup_{0\leqq u\leqq 1}\left|\int_t^{t+u} g(s,x)ds\right|\to 0 \qquad \text{as } t\to\infty;$$

(D–2) for some nondecreasing continuous function $b(\rho)$ defined on $[0,\infty)$ satisfying $b(0)=0$ and for each m $(0<m<r)$ there exists an absolutely diminishing function $h_m(t)$ such that

$$|g(t,x)-g(t,y)|\leqq b(|x-y|)+h_m(t)$$

for every $t\geqq 0$, $m\leqq|x|\leqq r$ and $m\leqq|y|\leqq r$.

Lemma 2.1. *Let* 0 *be EvUAS for* (E). *Suppose that for some* $r>0$, *each* $m\in(0,r)$, *and each* $\tau>0$, *there exists a nonincreasing continuous function* $F_{\tau,m}(\rho)$ *defined on* $[0,\infty)$ *satisfying* $F_{\tau,m}(\rho)\to 0$ *as* $\rho\to\infty$ *such that for each* $\tau_1\in(0,\tau]$, *each* $t_0\geqq\alpha(r)$ *and each solution* $y(\cdot)$ *of* (P–1) *satisfying* $|y(t_0)|<\delta(r)$ *and* $m\leqq|y(t)|\leqq r$ *for* $t_0\leqq t\leqq t_0+\tau_1$, *there exists a solution* $x(\cdot)$ *of* (E) *such that*

$$|x(t)-y(t)|\leqq F_{\tau,m}(t_0) \tag{2.1}$$

for all $t_0\leqq t\leqq t_0+\tau_1$. *Then* 0 *is EvUAS for* (P–1).

Remark. In the above statement $\alpha(r)$ and $\delta(r)$ come from Definition 2.1. The proof of this Lemma is found in [3].

3. Theorems. Now we state two Theorems.

Theorem A. *Let* $f(t,x)$ *satisfy the following condition;*

(i) $|f(t,x)-f(t,y)|\leqq L(t)|x-y|$ *for all* $t\geqq 0$ *and all* x, y *in* $S_r=\{x\in R^n:|x|\leqq r\}$, *and*

$$\int_s^t L(\tau)d\tau\leqq\chi(t-s) \qquad \textit{for all } t\geqq s\geqq 0,$$

where $\chi(\rho)$ *is a nondecreasing continuous function defined on* $[0,\infty)$ *such that* $\chi(0)=0$.

Let $g(t,x)$ *be diminishing. If* 0 *is EvUAS for* (E), *then* 0 *is EvUAS for* (P–1).

Theorem B. *Let* $f(t,x)$ *satisfy the following condition;*

(ii) $|f(t,x)-f(t,y)|\leqq L(t)|x-y|$ *for all* $t\geqq 0$ *and all* x, y *in* S_r, *and*

$$\sup_{t\geqq 0}\int_t^{t+1} L(s)ds=L<+\infty.$$

Let $g(t,x)$ *be absolutely diminishing and* $h(t)$ *be diminishing. If* 0 *is EvUAS for* (E), *then* 0 *is EvUAS for* (P–3).

4. Proof of Theorems. Before proving Theorems we state three lemmas without the proof.

Lemma 4.1. *If $g(t,x)$ is diminishing, then for each m $(0<m<r)$, it follows that*

$$\sup_{\substack{0\leqq u\leqq 1\\ m\leqq|x|\leqq r}}\left|\int_t^{t+u} g(s,x)ds\right|\to 0 \qquad \textit{as } t\to\infty.$$

Lemma 4.2. *Let $f(t,x)$ satisfy the condition* (ii). *If a solution $x(t)$ of* (E) *tends to 0 as $t\to\infty$, then $f(t,0)$ is diminishing.*

Lemma 4.3. *Let $f(t,x)$ satisfy the condition* (ii). *If $y(\cdot)$ is any solution of* (P–1) *such that $|y(t)|\to 0$ as $t\to\infty$, then the function $h(t)\equiv g(t,y(t))$ is diminishing.*

Proof of Theorem A. Let $0<m<r$. Define

$$E_m^*(t)=\sup\left|\int_t^{t+u} g(s,y(s))ds\right|,$$

where the supremum is evaluated with respect to all solutions $y(\cdot)$ of (P–1) satisfying

$$m\leqq|y(s)|\leqq r \qquad \text{for } t\leqq s\leqq t+u$$

and then with respect to all $0\leqq u\leqq 1$. We shall first prove that $E_m^*\to 0$ as $t\to\infty$.

Suppose $E_m^*(t)\not\to 0$ as $t\to\infty$. Then there exist $\varepsilon>0$, sequences $\{u_n\}$ and $\{t_n\}$, and a sequence of solutions $\{y_n(\cdot)\}$ of (P–1) on $[t_n, t_n+u_n]$ such that $0\leqq u_n\leqq 1$, $t_n\to\infty$, $m\leqq|y_n(s)|\leqq r$ for $t_n\leqq s\leqq t_n+u_n$, and

$$\left|\int_{t_n}^{t_n+u_n} g(s,y_n(s))ds\right|>\varepsilon.$$

Let w satisfy $0<w<1$, $\dfrac{1}{w}$ an integer, and

$$b(r\chi(w)+b(2r)w+w)<\varepsilon \tag{4.1}$$

where $b(\cdot)$ comes from Definition 2.3. Then

$$\left|\int_{t_n+jwu_n}^{t_n+(j+1)wu_n} g(s,y_n(s))ds\right|>\varepsilon w \tag{4.2}$$

for some integer j between 0 and $\dfrac{1}{w}-1$. Let $s_n=t_n+jwu_n$ and $v_n=t_n+(j+1)wu_n$ such that (4.2) holds. For $s_n\leqq t\leqq v_n$

$$\begin{aligned}|y_n(t)-y_n(s_n)|&\leqq\int_{s_n}^t|f(s,y_n(s))-f(s,0)|\,ds+\left|\int_{s_n}^t f(s,0)ds\right|\\&\quad+\int_{s_n}^t|g(s,y_n(s))-g(s,y_n(s_n))|\,ds+\left|\int_{s_n}^t g(s,y_n(s_n))ds\right|\\&\leqq r\chi(wu_n)+b(2r)wu_n+H_m(s_n),\end{aligned}$$

where

$$H_m(t)=\sup_{\substack{0\leqq u\leqq 1\\ m\leqq|x|\leqq r}}\left\{\left|\int_t^{t+u} f(s,0)ds\right|+\left|\int_t^{t+u} g(s,x)ds\right|+\int_t^{t+u} h_m(s)ds\right\}$$
$$\to 0 \text{ as } t\to\infty.$$

Then from (4.2) for every n

$$\varepsilon w < \left| \int_{s_n}^{v_n} g(s, y_n(s))ds \right| \leqq b(r\chi(wu_n) + b(2r)wu_n + H_m(s_n))wu_n + H_m(s_n).$$

Since $0 \leqq u_n \leqq 1$ and $H_m(s_n) \to 0$ and since

$$r\chi(wu_n) + b(2r)wu_n + H_m(s_n) \leqq r\chi(w) + b(2r)w + w$$

for large enough n, we must have

$$\varepsilon w \leqq b(r\chi(w) + b(2r)w + w)w.$$

This is a contradiction to (4.1). Thus $E_m^*(t) \to 0$ as $t \to \infty$.

Define

$$E_m(t) = \sup_{T \geqq t} E_m^*(T).$$

Then $E_m(t) \downarrow 0$ as $t \to \infty$. Let 0 be *EvUAS* for (E), so that we are given $\alpha(\cdot)$ and $\delta(\cdot)$. Let $\tau > 0$, let $\tau_1 \in (0, \tau]$, and let $t_0 \geqq \alpha(r)$. Let $y(\cdot)$ be any solution of (P–1) satisfying $|y(t_0)| < \delta(r)$ and $m \leqq |y(t)| \leqq r$ for $t_0 \leqq t \leqq t_0 + \tau_1$. Let $x(\cdot)$ be that solution of (E) such that $x(t_0) = y(t_0)$. Then if $t \in [t_0, t_0 + \tau_1]$, we have $|x(t)| \leqq r$ and hence

$$|x(t) - y(t)| \leqq \int_{t_0}^{t} L(s)\,|x(s) - y(s)|\,ds + (\tau + 1)E_m(t_0).$$

By Gronwall's Inequality, we obtain

$$|x(t) - y(t)| \leqq (\tau + 1)E_m(t_0) \exp \chi(\tau).$$

Since m was arbitrary, the hypotheses of Lemma 2.1 hold. The Lemma 2.1 will be used to complete the proof of Theorem A. Q.E.D.

The proof of Theorem B is analogous to the last half of the proof of Theorem A. The following corollary is an immediate consequence of Lemma 4.3 and Theorem B.

Corollary. *Let $f(t, x)$ satisfy the condition* (ii) *and* 0 *be EvUAS for* (E). *Then* 0 *is EvUAS for* (P–2) *if and only if $h(t)$ is diminishing.*

References

[1] W. A. Coppel: Stability and Asymptotic Behavior of Differential Equations. Heath, Boston (1965).

[2] A. Strauss and J. A. Yorke: Perturbation theorems for ordinary differential equations. J. Diff. Eqs., **3**, 15–30 (1967).

[3] ——: Perturbing Uniform Asymptotically Stable Nonlinear Systems. J. Diff. Eqs., **6**, 452–483 (1969).