

13. On a Sequence of Fourier Coefficients

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§ 1. Let $f(t)$ be a periodic function with period 2π and integrable (L) over $(-\pi, \pi)$. Let its Fourier series be

$$(1.1) \quad \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=0}^{\infty} A_n(t).$$

Then the conjugate series of (1.1) is

$$(1.2) \quad \sum_{n=1}^{\infty} (b_n \cos nt - a_n \sin nt) = \sum_{n=1}^{\infty} B_n(t).$$

Let $\{p_n\}$ be a sequence such that $P_n = \sum_{k=0}^n p_k \neq 0$ for $n=0, 1, 2, \dots$. A series $\sum_{n=0}^{\infty} a_n$ with its partial sum s_n is said to be summable (N, p_n) to sum s , if

$$\frac{1}{P_n} \sum_{k=0}^n p_{n-k} s_k \rightarrow s \quad \text{as } n \rightarrow \infty.$$

The $(N, p_n)(C, 1)$ method is obtained by superimposing the method (N, p_n) on the Cesàro means of order one.

Throughout this paper, let $\{p_n\}$ be a sequence such that $p_n \geq 0$, $p_n \downarrow$, $P_n \rightarrow \infty$, and we write

$$\psi(t) = f(x+t) - f(x-t) - l,$$

$$\Psi(t) = \int_0^t |\psi(u)| du$$

and $\tau = [1/t]$, where $[\lambda]$ is the integral part of λ .

§ 2. Varshney [9] proved that if

$$(2.1) \quad \Psi(t) = o(t/\log t^{-1}) \quad \text{as } t \rightarrow +0,$$

then the sequence $\{nB_n(x)\}$ is summable $(N, 1/(n+1))(C, 1)$ to l/π . This was generalized by Sharma [6], Singhal [8] and Dikshit [1], respectively, as follows.

Theorem A (Sharma [6]). *If*

$$(2.2) \quad \Psi(t) = o(t) \quad \text{as } t \rightarrow +0,$$

and, for some fixed δ , $0 < \delta < 1$,

$$(2.3) \quad \int_t^\delta \frac{|\psi(u)|}{u} \log \frac{1}{u} du = o(\log t^{-1}) \quad \text{as } t \rightarrow +0,$$

then the sequence $\{nB_n(x)\}$ is summable $(N, 1/(n+1))(C, 1)$ to l/π .

Remark 1. (2.3) implies (2.2), because

$$\begin{aligned}
\Psi(t) &= \int_0^t |\psi(u)| du = \int_0^t \frac{|\psi(u)|}{u} \log \frac{1}{u} \frac{u}{\log u^{-1}} du \\
&= - \left[\frac{u}{\log u^{-1}} \int_u^s \frac{|\psi(x)|}{x} \log \frac{1}{x} dx \right]_0^t \\
&\quad + \int_0^t \frac{\log u^{-1} + 1}{(\log u^{-1})^2} du \int_u^s \frac{|\psi(x)|}{x} \log \frac{1}{x} dx \\
&= o(t).
\end{aligned}$$

Therefore, we see that *the assumption (2.2) is superfluous.*

Theorem B (Singhal [8]). *Let $\alpha(t)$ be a positive non-decreasing function such that $\log n = O(\alpha(P_n))$, as $n \rightarrow \infty$, and $t/\alpha(t)$ is non-decreasing. If*

$$(2.4) \quad \Psi(t) = o(t/\alpha(P_n)) \quad \text{as } t \rightarrow +0,$$

then the sequence $\{nB_n(x)\}$ is summable $(N, p_n)(C, 1)$ to l/π .

Remark 2. We see, from the proof of Theorem 2 in [3], that *the assumption on monotone of $\alpha(t)$ is superfluous.*

Remark 3. Singhal has written his theorem with

$$\Psi(t) = o(t/\log t^{-1}) \quad \text{as } t \rightarrow +0,$$

instead of (2.4). But he has proved his theorem under the condition (2.4).

Theorem C (Dikshit [1]). *Let $\beta(t)$ be a non-negative non-decreasing function such that $\beta(n) \log n = O(P_n)$, as $n \rightarrow \infty$. If*

$$(2.5) \quad \Psi(t) = o(t\beta(1/t)/P_n) \quad \text{as } t \rightarrow +0,$$

then the sequence $\{nB_n(x)\}$ is summable $(N, p_n)(C, 1)$ to l/π .

Remark 4. If Theorem C holds, then Theorem B also holds and conversely, when $\beta(1/t)/\beta(\tau) = O(1)$ as $t \rightarrow +0$, if Theorem B holds, then Theorem C also holds. (See [3], Theorem 2.)

Theorem D (Dikshit [1]). *Let $\gamma(t)$ be a non-negative function such that (i) $\{\gamma(t)/(t \log t)\}$ is monotonic, (ii) $\gamma(t) = O(\log t)$, $t \rightarrow \infty$ and (iii)*

$$(2.6) \quad \sum_{k=2}^n \gamma(k) P_k / (k \log k) = O(P_n), \quad \text{as } n \rightarrow \infty.$$

If

$$(2.7) \quad \Psi(t) = o(t\gamma(1/t)/\log t^{-1}) \quad \text{as } t \rightarrow +0,$$

then the sequence $\{nB_n(x)\}$ is summable $(N, p_n)(C, 1)$ to l/π .

Remark 5. After the proof of Theorem 2 below, we see that *the condition (ii) is superfluous.* And Theorem in which $\gamma(t) = 1$ is due to Singh [7].

The main result of this paper is the following theorem which includes all the results mentioned above.

Theorem 1. *Let $p(t)$ be monotone non-increasing and positive for $t \geq 0$. Let $p_n = p(n)$ and let*

$$(2.8) \quad P(t) = \int_0^t p(u) du \rightarrow \infty \quad \text{as } t \rightarrow +\infty.$$

If, for some fixed δ , $0 < \delta < 1$,

$$(2.9) \quad \int_{1/n}^{\delta} \Psi(t) \frac{d}{dt} \frac{P(1/t)}{t} dt = o(P_n) \quad \text{as } n \rightarrow \infty,$$

then the sequence $\{nB_n(x)\}$ is summable $(N, p_n)(C, 1)$ to l/π .

Theorem 2. *If Theorem 1 holds, then Theorems A, B, C and D also hold.*

Corollary 1. *Let $\alpha(t)$ be a positive function such that*

$$\int_{1/n}^{\delta} \frac{P_{\tau}}{\alpha(P_{\tau})} \frac{1}{t} dt = O(P_n), \quad 0 < \delta < 1.$$

If

$$\Psi(t) = o(t/\alpha(P_{\tau})) \quad \text{as } t \rightarrow +0,$$

then the sequence $\{nB_n(x)\}$ is summable $(N, p_n)(C, 1)$ to l/π .

This is contained in the result of Theorem 1 (see [3], pp. 305-306), but this includes Theorem B.

Corollary 2. *Let $\beta(t)$ be a positive integrable function such that*

$$\int_{\eta}^n u^{-1} \beta(u) du = O(P_n) \quad \text{as } n \rightarrow \infty, \quad \text{for any fixed } \eta > 0.$$

If

$$\Psi(t) = o(t\beta(1/t)/P_{\tau}) \quad \text{as } t \rightarrow +0,$$

then the sequence $\{nB_n(x)\}$ is summable $(N, p_n)(C, 1)$ to l/π .

This is also contained in the result of Theorem 1, but this includes Theorem C (see [2], p. 450). Further we see that this includes Corollary 1 and conversely, when $\beta(1/t) \sim \beta(\tau)$ as $t \rightarrow +0$, Corollary 1 includes Corollary 2.

§ 3. Proof of Theorem 1. From the method of Mohanty and Nanda [4], we have

$$\frac{1}{n} \sum_{k=1}^n kB_k(x) - \frac{l}{\pi} = \frac{1}{\pi} \int_0^{\delta} \psi(t) \left\{ \frac{\sin nt}{nt^2} - \frac{\cos nt}{t} \right\} dt + o(1),$$

by Riemann-Lebesgue's theorem. Since the (N, p_n) method is regular, in order to prove our theorem it is sufficient to show that

$$(3.1) \quad \begin{aligned} & \frac{1}{P_n} \sum_{k=1}^n p_{n-k} \frac{1}{\pi} \int_0^{\delta} \psi(t) \left\{ \frac{\sin kt}{kt^2} - \frac{\cos kt}{t} \right\} dt \\ & = \int_0^{\delta} \psi(t) N_n(t) dt = o(1), \end{aligned}$$

where

$$(3.2) \quad N_n(t) = \frac{1}{\pi P_n} \sum_{k=1}^n p_{n-k} \left\{ \frac{\sin kt}{kt^2} - \frac{\cos kt}{t} \right\}$$

$$(3.3) \quad = \begin{cases} O(n) & (0 < t \leq 1/n) \\ O(P_{\tau}/tP_n) & (0 < t \leq \pi) \end{cases}.$$

(For these estimations, see Singh's [7], p. 441 or Singhal's [8], pp. 273-274.) Now we write

$$\int_0^{\delta} \psi(t) N_n(t) dt = \left(\int_0^{1/n} + \int_{1/n}^{\delta} \right) \psi(t) N_n(t) dt = I_1 + I_2, \text{ say.}$$

From Rajagopal's lemma ([5], Lemma (a)), (2.8) and (2.9) together imply

$$\Psi(t) = o(t) \quad \text{as } t \rightarrow +0.$$

Hence, by (3.2), we have

$$I_1 = O(n) \left(\int_0^{1/n} |\psi(t)| dt \right) = o(1).$$

Next, by (3.3) and (2.9), we have

$$\begin{aligned} I_2 &= O \left(\int_{1/n}^{\infty} |\psi(t)| \frac{P_{\tau}}{tP_n} dt \right) = O \left(\int_{1/n}^{\infty} |\psi(t)| \frac{P(1/t)}{tP_n} dt \right) \\ &= O \left(\left[\Psi(t) \frac{P(1/t)}{tP_n} \right]_{1/n}^{\infty} \right) + O \left(\frac{1}{P_n} \int_{1/n}^{\infty} \Psi(t) \left| \frac{d}{dt} \frac{P(1/t)}{t} \right| dt \right) \\ &= o(1). \end{aligned}$$

Therefore we have (3.1) and the proof of Theorem 1 is complete.

§ 4. Proof of Theorem 2. For the proof, since, by Remark 4, Theorem C includes Theorem B, it is sufficient to prove that the conditions of Theorems A, C and D, respectively, imply the one of Theorem 1. We define a function $p(t)$ by

$$p(t) = p_n \quad \text{for } n \leq t < n+1, \quad n = 0, 1, 2, \dots$$

Further define a function $P(t)$ as in (2.8). Then the function $p(t)$ is monotone non-increasing and positive for $t \geq 0$. And $P(t) \rightarrow \infty$ as $t \rightarrow \infty$. Furthermore, we get, $P_{\tau} \sim P(1/t)$ and $1/tp(1/t) \sim P(1/t)$ as $t \rightarrow +0$. Here, we want to show that the condition (2.9) is satisfied under the assumption.

Concerning Theorem A, since $(P(1/t)/t)' = O(P(1/t)/t^2) = O((\log t^{-1})/t^2)$, we have

$$\begin{aligned} \int_{1/n}^{\infty} \Psi(t) \frac{d}{dt} \frac{P(1/t)}{t} dt &= O \left(\int_{1/n}^{\infty} \Psi(t) \frac{\log t^{-1}}{t^2} dt \right) \\ &= O \left(\int_{1/n}^{\infty} \frac{\log t^{-1}}{t^2} dt \int_0^t |\psi(u)| du \right) \\ &= O \left(\int_{1/n}^{\infty} |\psi(u)| du \int_u^{\infty} \frac{\log t^{-1}}{t^2} dt + \int_0^{1/n} |\psi(u)| du \int_{1/n}^{\infty} \frac{\log t^{-1}}{t^2} dt \right) \\ &= O \left(\int_{1/n}^{\infty} |\psi(u)| \log \frac{1}{u} du \int_u^{\infty} \frac{1}{t^2} dt \right) + O \left(\int_0^{1/n} |\psi(u)| du (\log n) \int_{1/n}^{\infty} \frac{1}{t^2} dt \right) \\ &= O \left(\int_{1/n}^{\infty} \frac{|\psi(u)|}{u} \log \frac{1}{u} du \right) + O(\Psi(1/n)n \log n) \\ &= o(\log n) \\ &= o(P_n), \end{aligned}$$

from (2.3) and (2.2). Thus the condition (2.9) is satisfied.

Next, concerning Theorem C, we have, from (2.5),

$$\begin{aligned} \int_{1/n}^{\infty} \Psi(t) \frac{d}{dt} \frac{P(1/t)}{t} dt &= O \left(\int_{1/n}^{\infty} \Psi(t) \frac{P(1/t)}{t^2} dt \right) \\ &= o \left(\int_{1/n}^{\infty} \frac{t\beta(1/t)}{P_{\tau}} \frac{P(1/t)}{t^2} dt \right) \end{aligned}$$

$$\begin{aligned}
&= o\left(\int_{1/n}^{\sigma} \frac{1}{t} \beta\left(\frac{1}{t}\right) dt\right) \\
&= o(\beta(n) \log n) \\
&= o(P_n),
\end{aligned}$$

since the function $\beta(t)$ is non-decreasing. Thus the condition (2.9) is satisfied.

Lastly, concerning Theorem D, we have, from (2.7) and (2.6),

$$\begin{aligned}
\int_{1/n}^{\sigma} \Psi(t) \frac{d}{dt} \frac{P(1/t)}{t} dt &= O\left(\int_{1/n}^{\sigma} \Psi(t) \frac{P(1/t)}{t^2} dt\right) \\
&= o\left(\int_{1/n}^{\sigma} \frac{t\gamma(1/t)}{\log t^{-1}} \frac{P(1/t)}{t^2} dt\right) \\
&= o\left(\int_{1/n}^{\sigma} \frac{\gamma(1/t)P(1/t)}{t \log t^{-1}} dt\right) \\
&= o\left(\int_{1/\sigma}^n \frac{\gamma(u)P(u)}{u \log u} du\right) \\
&= o\left(\sum_{k=2}^n \frac{\gamma(k)P_k}{k \log k}\right) \\
&= o(P_n),
\end{aligned}$$

since the function $\gamma(t)/(t \log t)$ is monotone. Thus the condition (2.9) is satisfied.

Therefore, the proof of Theorem 2 is complete.

§ 5. The proof of Theorem 2 together with known theorems ([5], Theorem 1 and [3], Theorem 1) shows that the following theorems are obtained. In the rest of this paper, we use the following notations.

$$g(t) = \frac{1}{2}\{f(x+t) + f(x-t) - 2f(x)\}, \quad G(t) = \int_0^t |g(u)| du,$$

$$h(t) = \frac{1}{2}\{f(x+t) - f(x-t)\}, \quad H(t) = \int_0^t |h(u)| du.$$

Theorem 3. Under the conditions of Theorem A, if we replace $\psi(t)$ by $g(t)$ and $\Psi(t)$ by $G(t)$, then the series $\sum_{n=0}^{\infty} A_n(x)$ is summable $(N, 1/(n+1))$ to $f(x)$.

Theorem 4. Under the conditions of Theorem A, if we replace $\psi(t)$ by $h(t)$ and $\Psi(t)$ by $H(t)$, then the series $\sum_{n=1}^{\infty} B_n(x)$ is summable $(N, 1/(n+1))$ to

$$(5.1) \quad \bar{f}(x) = \frac{1}{\pi} \int_0^{\pi} h(u) \cot \frac{u}{2} du$$

provided that this integral exists as a Cauchy integral at origin.

Theorem 5. Under the conditions of Theorem D, if we replace $\Psi(t)$ by $G(t)$, then the series $\sum_{n=0}^{\infty} A_n(x)$ is summable (N, p_n) to $f(x)$.

Theorem 6. Under the conditions of Theorem D, if we replace $\Psi(t)$

by $H(t)$, then the series $\sum_{n=1}^{\infty} B_n(x)$ is summable (N, p_n) to $\tilde{f}(x)$ provided that the integral in (5.1) exists as a Cauchy integral at origin.

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