9. *On the Completions of Maps*

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(Comm. by Kenjiro SHODA, M. J. A., Jan. 12, 1974)

In this paper all spaces are assumed to be completely regular T_2. Let f be a continuous map from a space X into a space Y. As is well known, there exists its extension $\beta(f):\beta(X)\to\beta(Y)$, where $\beta(S)$ denotes the Stone-Čech compactification of a space S. Furthermore, it is known that $\beta(f)$ carries $\mu(X)$ into $\mu(Y)$ and $\upsilon(X)$ into $\upsilon(Y)$ ([14], [3]), where $\mu(X)$ is the topological completion of X (that is, the completion of X with respect to its finest uniformity μ) and $\upsilon(X)$ is the realcompactification of X. We denote the restriction maps $\beta(f)|\mu(X)$ and $\beta(f)|\upsilon(X)$ by $\mu(f)$ and $\upsilon(f)$ respectively.

The purpose of this paper is to study the relations between f and $\mu(f)$ (or $\upsilon(f)$).

We note first that $\mu(f):\mu(X)\to\mu(Y)$ and $\upsilon(f):\upsilon(X)\to\upsilon(Y)$ are not necessarily perfect even if $f:X\to Y$ is perfect. A continuous map f from a space X onto a space Y is called a quasi-perfect (perfect) map if f is a closed map such that $f^{-1}(y)$ is countably compact (resp. compact) for each $y\in Y$.

Example. Let Y be a pseudo-compact space such that the preimage X of Y under a perfect map f is not pseudo-compact ([4, Example 4.2]). Then both $\mu(f):\mu(X)\to\mu(Y)$ and $\upsilon(f):\upsilon(X)\to\upsilon(Y)$ are not perfect, since $\mu(X)$ and $\upsilon(X)$ are not compact, while $\mu(Y)$ and $\upsilon(Y)$ are compact (cf. [14], [3]).

In view of these results, it is significant to study under what conditions $\mu(f)$ (or $\upsilon(f)$) is perfect.

Theorem 1. *If $f:X\to Y$ is an open quasi-perfect map, then $\mu(f):\mu(X)\to\mu(Y)$ and $\upsilon(f):\upsilon(X)\to\upsilon(Y)$ are open perfect.*

To prove this theorem, we use the following lemmas.

Lemma 2 (Zenor [17]). *Let $C(X)$ be the space of all the non-empty compact sets in a space X with the finite topology. If X is topologically complete, so is $C(X)$.*

The finite topology of $C(X)$ is defined as follows: For any finite number of open sets $\{U_1,\cdots,U_n\}$ of X, we set $[U_1,\cdots,U_n]=\{K\in C(X)|K\subset\bigcup_{i=1}^{n}U_i, K\cap U_i\neq\emptyset$ for $i=1,\cdots,n\}$. As an open base of $C(X)$ we take all such sets. It is well known that if X is completely regular then so is $C(X)$ (Michael [12]).

Lemma 3. *If $f: X \to Y$ is an open quasi-perfect map, then $\varphi: Y \to C(\mu(X))$ and $\varphi^*: Y \to C(\upsilon(X))$ are continuous, where $\varphi(y) = \mathrm{cl}_{\mu(X)} f^{-1}(y)$ and $\varphi^*(y) = \mathrm{cl}_{\upsilon(X)} f^{-1}(y)$ for each $y \in Y$.*

Hoshina [5] proved the continuity of $\varphi: Y \to C(\mu(X))$, and the continuity of $\varphi^*: Y \to C(\upsilon(X))$ is similarly proved.

Proof of Theorem 1. We note first that a surjective map $g: X \to Y$ is perfect if and only if any filter base $\{F_\alpha\}$ in X such that $\{g(F_\alpha)\}$ has a cluster point in Y has a cluster point in X. Now we prove the theorem for the case of $\mu(f)$, since the case of $\upsilon(f)$ is similarly proved. Let $\mathfrak{F} = \{F_\alpha\}$ be a filter base in $\mu(X)$ such that $\{\mu(f)(F_\alpha)\}$ has a cluster point v in $\mu(Y)$. Let us put

$$\mathfrak{G} = \{G_\gamma \mid v \in G_\gamma, G_\gamma : \text{open in } \mu(Y)\},$$
$$\mathfrak{G}_Y = \{H_\gamma \mid H_\gamma = G_\gamma \cap Y, G_\gamma \in \mathfrak{G}\}.$$

Then $\mathfrak{G}_Y$ is a Cauchy filter base in Y with respect to μ, and it converges to v in $\mu(Y)$. Since $\varphi: Y \to C(\mu(X))$ is continuous by Lemma 3, $\{\varphi(H_\gamma)\}$ is a Cauchy filter base in $C(\mu(X))$ with respect to the finest uniformity, and hence by Lemma 2 $\{\varphi(H_\gamma)\}$ converges to some $K \in C(\mu(X))$. Suppose that $(\bigcap \mathrm{cl}_{\mu(X)} F_\alpha) \cap K = \emptyset$. Then for each point u of K there exists $F_{\alpha(u)}$ of $\mathfrak{F}$ such that $u \in \mu(X) - \mathrm{cl}_{\mu(X)} F_{\alpha(u)}$. Therefore there exists a finite number of points $\{u_1, \cdots, u_n\}$ of K such that

$$K \subset \bigcup_{i=1}^{n} (\mu(X) - \mathrm{cl}_{\mu(X)} F_{\alpha(u_i)}),$$

since K is compact. Let F_β be an element of $\mathfrak{F}$ such that $F_\beta \subset F_{\alpha(u_i)}$, $i = 1, \cdots, n$. Then we have

$$\bigcup_{i=1}^{n} (\mu(X) - \mathrm{cl}_{\mu(X)} F_{\alpha(u_i)}) \cap F_\beta = \emptyset.$$

Let O be a regularly open set in $\mu(X)$ such that

$$K \subset O \subset \mathrm{cl}_{\mu(X)} O \subset \bigcup_{i=1}^{n} (\mu(X) - \mathrm{cl}_{\mu(X)} F_{\alpha(u_i)}).$$

Since $\{f^{-1}(H_\gamma)\}$ converges to K in $C(\mu(X))$, we have $f^{-1}(H_\gamma) \subset O$ for some $H_\gamma \in \mathfrak{G}_Y$, and hence $\mu(f)^{-1}(G_\gamma) \subset O$. This shows that $\mu(f)^{-1}(G_\gamma) \cap F_\beta = \emptyset$, that is, $G_\gamma \cap \mu(f)(F_\beta) = \emptyset$, which is a contradiction. Therefore we have $(\bigcap \mathrm{cl}_{\mu(X)} F) \cap K \neq \emptyset$. Consequently $\mathfrak{F}$ has a cluster point in $\mu(X)$. Moreover from the fact mentioned above it is easily seen that $\mu(f): \mu(X) \to \mu(Y)$ is surjective. Hence $\mu(f): \mu(X) \to \mu(Y)$ is perfect. Finally, by [10, Theorem 4.4], $\beta(f): \beta(X) \to \beta(Y)$ is an open map. Therefore it follows that $\mu(f)$ is an open map. Thus we complete the proof.

Corollary 4. *Let $f: X \to Y$ be an open perfect map. Then the following statements are valid.*

(a) *Y is topologically complete if and only if X is topologically complete.*

(b) *Y is realcompact if and only if X is realcompact* (*Frolík* [2]).

This corollary follows from Theorem 1 and the fact that the preimage of a topologically complete (realcompact) space under a perfect map is also topologically complete (resp. realcompact).

A continuous map f from a space X onto a space Y is called a WZ-map (Isiwata [10]) if $\beta(f)^{-1}(y)=\mathrm{cl}_{\beta(X)} f^{-1}(y)$ for each $y\in Y$. Every closed map is a WZ-map. The following is a slight generalization of Theorem 1.

Theorem 5. *If $f: X\to Y$ is an open WZ-map such that $f^{-1}(y)$ is relatively pseudo-compact for each $y\in Y$, then $\mu(f): \mu(X)\to\mu(Y)$ and $\upsilon(f): \upsilon(X)\to\upsilon(Y)$ are open perfect.*

Proof. Let $X_0=\beta(f)^{-1}(Y)$. Since $\beta(f)^{-1}(y)=\mathrm{cl}_{\beta(X)} f^{-1}(y)$ and $\mathrm{cl}_{\mu(X)} f^{-1}(y)$ is compact, we have $X\subset X_0\subset\mu(X)\subset\upsilon(X)$. Hence it follows that $\mu(X_0)=\mu(X)$ and $\upsilon(X_0)=\upsilon(X)$ ([14], [3]). On the other hand, $\mu(f): \mu(X_0)\to\mu(Y)$ and $\upsilon(f): \upsilon(X_0)\to\upsilon(Y)$ are open perfect by Theorem 1, since $\beta(f)|X_0: X_0\to Y$ is an open perfect map. Thus the theorem holds.

Corollary 6. *Let $f: X\to Y$ be an open WZ-map such that $f^{-1}(y)$ is relatively pseudo-compact for each $y\in Y$. Then the following statements are valid.*

(a) *Y is pseudo-compact if and only if X is pseudo-compact.*

(b) *Y is pseudo-paracompact (pseudo-Lindelöf) if and only if X is pseudo-paracompact (resp. pseudo-Lindelöf).*

Following Morita [14], a space X is said to be pseudo-paracompact (resp. Lindelöf) if $\mu(X)$ is paracompact (resp. Lindelöf). In Corollary 6, (a) was proved by Isiwata [10] as a generalization of a theorem of Okuyama and Hanai [16], and the 'only-if' part of (b) was proved by Hoshina [5].

Concerning a (not necessarily open) quasi-perfect map, Morita [14] proved the following: If f is a quasi-perfect map from an M-space X onto an M-space Y, then $\mu(f): \mu(X)\to\mu(Y)$ is a perfect map. As a generalization of this result, we can prove the following theorem.

Theorem 7. *Let X and Y be the spaces each of which is the pre-image of a topologically complete space under a quasi-perfect map. If $f: X\to Y$ is a quasi-perfect map, then $\mu(f): \mu(X)\to\mu(Y)$ is a perfect map.*

To prove Theorem 7, we use the following lemmas.

Lemma 8 (Ishii [9]). *If f is a quasi-perfect map from a space X onto a topologically complete space Y, then $\mu(f): \mu(X)\to Y$ is perfect.*

Lemma 9 (Kljušin [6]). *Let $f: X\to Y$ and $g: Y\to Z$ be surjective. If $h=g\circ f: X\to Z$ is perfect, then f and g are perfect.*

Proof of Theorem 7. Let $g: Y\to Z$ be a quasi-perfect map from Y onto a topologically complete space Z. Then $h=g\circ f: X\to Z$ is a quasi-perfect map, and hence $\mu(h): \mu(X)\to Z$ is a perfect map by Lemma 8. Let $Y_0=\mu(f)(\mu(X))$. Since $\mu(h)=\mu(g\circ f)=\mu(g)\circ\mu(f)$, $\mu(g)|Y_0: Y_0\to Z$ is perfect by Lemma 9. Hence it follows that Y_0 is topologically complete, which implies that $Y_0=\mu(Y)$. Therefore $\mu(f): \mu(X)\to\mu(Y)$ is perfect by Lemma 9. Thus we complete the proof.

Corollary 10. *Let X and Y be the spaces each of which is the preimage of a topologically complete space under a quasi-perfect map, and let $f: X\to Y$ be a quasi-perfect map. Then Y is pseudo-paracompact (pseudo-Lindelöf) if and only if X is pseudo-paracompact (resp. pseudo-Lindelöf).*

Remark. By Lemma 8, a space X is the preimage of a paracompact space under a quasi-perfect map if and only if X is a pseudo-paracompact space which is the preimage of a topologically complete space under a quasi-perfect map.

Applying Theorem 7, we can prove the following theorem.

Theorem 11. *Let Y be an M^*-space* ([7]). *Then the following statements are equivalent.*

(a) *Y is the preimage of a topologically complete space under a quasi-perfect map.*

(b) *Y is an M-space.*

Proof. Since (b)→(a) is obvious, we shall prove (a)→(b). Since Y is an M^*-space, there exists a perfect map f from an M-space X onto Y by Nagata's theorem [15]. Hence by Theorem 7 $\mu(f): \mu(X)\to\mu(Y)$ is a perfect map. Since $\mu(X)$ is a paracompact M-space by Morita's theorem [14] and the image of a paracompact M-space under a perfect map is also a paracompact M-space (cf. Fillipov [1], Ishii [7], [8] and Morita [13]), $\mu(Y)$ is a paracompact M-space. This implies that Y is an M'-space ([14]). Since each M^*-space is countably paracompact ([7]), Y is an M-space ([11]). Thus we complete the proof.

We note that Theorem 11 is also deduced directly from Lemma 8.

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