4. *A Proof of Ehrenpreis' Fundamental Principle in Hyperfunctions*

By Toshio OSHIMA
Department of Mathematics, University of Tokyo

(Comm. by Kôsaku YOSIDA, M. J. A., Jan. 12, 1974)

1. Introduction. The purpose of this note is to give another easier proof of the theorem of integral representation for hyperfunction solutions of linear partial differential equations with constant coefficients which was first formulated and proved in Kaneko [3], [4].

Most results in the general theory of systems of linear partial differential equations with constant coefficients are deduced from Ehrenpreis' Fundamental Principle (cf. Ehrenpreis [1], [2] and Palamodov [6]), which says the following:

Let $\mathcal{P}$ denote the ring of linear partial differential operators with constant coefficients in n variables. Given an $r_1 \times r_0$ matrix $P(D_x)$ with elements in $\mathcal{P}$, we can define a multiplicity variety $\mathfrak{V}$ which is a set of finite pairs of irreducible affine algebraic varieties V_λ in $\boldsymbol{C}^n$ and row vectors $\partial_\lambda(\zeta, D_\zeta)$ of length r_0 whose elements are differential operators in $\boldsymbol{C}^n$ with polynomial coefficients (which are called noetherian operators in Palamodov [6]). Let $\mathcal{F}$ be a certain function space of $\mathcal{P}$-module. Then every kernel u of the map $P(D_x): \mathcal{F}^{r_0} \to \mathcal{F}^{r_1}$ can be expressed in the form

$$u(x)=\sum_\lambda \int_{V_\lambda} {}^t\partial_\lambda(\zeta, D_\zeta) \exp \langle \sqrt{-1}\, x, \zeta \rangle \, d\mu_\lambda(\zeta), \tag{1}$$

where each μ_λ is a measure with support in V_λ which satisfies some growth conditions at infinity determined by $\mathcal{F}$. The integral converges in the topology of $\mathcal{F}$.

When $\mathcal{F}$ is the space of distributions or infinitely differentiable functions on a convex domain in $\boldsymbol{R}^n$ or holomorphic functions on a convex domain in $\boldsymbol{C}^n$, the above statement is proved by Ehrenpreis [2] and Palamodov [6]. In case $\mathcal{F}$ is the space of hyperfunctions $\mathcal{B}(\Omega)$ on a convex domain Ω in $\boldsymbol{R}^n$, the measures in (1) satisfy

$$\int_{V_\lambda} \exp(-\varepsilon|\zeta| + H_K(\zeta))\, |d\mu_\lambda(\zeta)| < \infty, \qquad \text{for } \forall \varepsilon > 0,\ \forall K \Subset \Omega, \tag{2}$$

where $H_K(\zeta) = \sup_{x \in K} \operatorname{Re} \langle \sqrt{-1}\, x, \zeta \rangle$. The integral is considered in the sense of hyperfunctions. (See Kaneko [3] or the proof below.) We give a proof in this case using the result in the case when $\mathcal{F}$ is the space of holomorphic functions.

2. Proof. Set $U=\{z \in \boldsymbol{C}^n ; \operatorname{Re} z \equiv (\operatorname{Re} z_1, \cdots, \operatorname{Re} z_n) \in \Omega\}$ and $U_i = \{z \in U ; \operatorname{Im} z_i \neq 0\}$. Since U and U_i are Stein open sets in $\boldsymbol{C}^n$, Leray's theorem on cohomology groups with respect to the covering system $\mathcal{U}=\{U, U_1, \cdots, U_n\}$ and $\mathcal{U}'=\{U_1, \cdots, U_n\}$ says that $H^n(\mathcal{U}, \mathcal{U}', \mathcal{O})=H^n_\Omega(U, \mathcal{O})$, which equals $\mathcal{B}(\Omega)$ by definition. Here $\mathcal{O}$ denotes the sheaf of germs of holomorphic functions over C^n.

Let

$$0 \longleftarrow \mathcal{P}^{r_0}/{}^tP(D_x)\mathcal{P}^{r_1} \longleftarrow \mathcal{P}^{r_0} \xleftarrow{{}^tP(D_x)} \mathcal{P}^{r_1} \xleftarrow{{}^tP_1(D_x)} \mathcal{P}^{r_2} \longleftarrow \cdots$$

be a free resolution. Then we can define the double complex

$$K^{pq}=C^p(\mathcal{U}, \mathcal{U}', \mathcal{O}^{r_q}), \qquad p\geqslant 0,\ q\geqslant 0,$$

with the following differentials:

$$d' : C^p(\mathcal{U}, \mathcal{U}', \mathcal{O}^{r_q}) \to C^{p+1}(\mathcal{U}, \mathcal{U}', \mathcal{O}^{r_q})$$

is the natural coboundary operator and

$$d'' : C^p(\mathcal{U}, \mathcal{U}', \mathcal{O}^{r_q}) \to C^p(\mathcal{U}, \mathcal{U}', \mathcal{O}^{r_{q+1}})$$

is the operator defined by $(-1)^pP_q(D_z)$. Then we have

$$'E_1^{pq}=\begin{cases} C^p(\mathcal{U}, \mathcal{U}', \mathcal{O}_P^{r_0}) & \text{if } p\leqslant n \text{ and } q=0, \\ 0 \text{ if } p>n \text{ or } q\neq 0, \end{cases}$$

which follows from the solvability in a convex domain in C^n (cf. Theorem 2 in Komatsu [5]). Here $\mathcal{O}_P^{r_0}$ denotes the solution sheaf $\mathcal{H}om_{\mathcal{P}}(\mathcal{P}^{r_0}/{}^tP(D_z)\mathcal{P}^{r_1}, \mathcal{O})$. On the other hand, we have

$$''E_1^{pq}=\begin{cases} H^n(\mathcal{U}, \mathcal{U}', \mathcal{O}^{r_q})=\mathcal{B}(\Omega)^{r_q} & \text{if } p=n, \\ H^p(\mathcal{U}, \mathcal{U}', \mathcal{O}^{r_q})=0 & \text{if } p\neq n. \end{cases}$$

This follows from the fundamental theorem in hyperfunctions. Hence we have $E^{pq}=0$ when $p\neq n$ or $q\neq 0$ and have

$$\begin{aligned} E^{n0} &= C^n(\mathcal{U}, \mathcal{U}', \mathcal{O}_P^{r_0})/d'C^{n-1}(\mathcal{U}, \mathcal{U}', \mathcal{O}_P^{r_0}) \\ &= \ker (P(D_x) : \mathcal{B}(\Omega)^{r_0} \to \mathcal{B}(\Omega)^{r_1}). \end{aligned}$$

Put $I=\{(\sigma_1, \cdots, \sigma_n); \sigma_i=-1 \text{ or } 1\}$ and $W_\sigma=\{z\in U; \sigma_i \operatorname{Im} z_i>0, 1\leqslant i\leqslant n\}$ for $\sigma\in I$. Then the element of $C^n(\mathcal{U}, \mathcal{U}', \mathcal{O}_P^{r_0})$ is the set of 2^n tuples of vectors of holomorphic functions $\{F_\sigma(z)\}_{\sigma\in I}$, where every $F_\sigma(z)$ is defined on W_σ and satisfies $P(D_z)F_\sigma(z)=0$. Therefore any $u\in\mathcal{B}(\Omega)^{r_0}$ satisfying $P(D_x)u=0$ is represented by the cohomology class of the above $\{F_\sigma(z)\}_{\sigma\in I}$. Ehrenpreis [2] and Palamodov [6] show that $F_\sigma(z)$ is expressed in the form

$$\operatorname{sign}\sigma\cdot F_\sigma(z)=\sum_\lambda \int_{V_\lambda} {}^t\partial_\lambda(\zeta, D_\zeta) \exp\langle\sqrt{-1}\,z, \zeta\rangle\, d\mu_\lambda^\sigma(\zeta),$$

where we set $\operatorname{sign}\sigma=\prod_{i=1}^n \sigma_i$ and the measure μ_λ^σ on V_λ satisfies

$$\int_{V_\lambda} \exp H_L(\zeta)\cdot|d\mu_\lambda^\sigma(\zeta)|<\infty \qquad \text{for } {}^\forall L\subset W_\sigma.$$

Given $\varepsilon>0$ and $K\subset\Omega$, we set $L=\{z\in C^n; \operatorname{Re} z\in K, \sigma_i \operatorname{Im} z_i=\varepsilon, \text{ for } 1\leqslant i\leqslant n\}\subset W_\sigma$. Then we have

$$\sup_{x\in K} \operatorname{Re}\langle\sqrt{-1}\,x, \zeta\rangle\leqslant\sup_{z\in L} \operatorname{Re}\langle\sqrt{-1}\,z, \zeta\rangle+\varepsilon|\zeta|.$$

Hence it is clear that each μ_λ^σ satisfies (2).

Now we mention the meaning of the integral (1) in the sense of hyperfunctions: We can write $\mu_\lambda = \sum_{\sigma \in I} \mu_{\lambda,\sigma}$ such that each $\mu_{\lambda,\sigma}$ also satisfies (2) and has its support in the set $\Gamma_\sigma = \{\zeta \in C^n ; \operatorname{Re}(\sigma_i \zeta_i) \geqslant 0$ for $1 \leqslant i \leqslant n\}$. We put

$$G_\sigma(z) = \operatorname{sign} \sigma \sum_\lambda \int_{V_\lambda} {}^t\partial_\lambda(\zeta, D_\zeta) \exp \langle \sqrt{-1}\, z, \zeta \rangle \, d\mu_{\lambda,\sigma}(\zeta).$$

Then $G_\sigma(z)$ is holomorphic in W_σ and (1) represents a vector of hyperfunctions $u(x) \in \mathcal{B}(\Omega)^{r_0}$ as the cohomology class of $\{G_\sigma(z)\}_{\sigma \in I}$. We remark that $P(D_x)u(x) = 0$ because $P(D_z)G_\sigma(z) = 0$.

We set $\mu_\lambda = \sum_{\sigma \in I} \mu_\lambda^\sigma$. If we show that $\{F_\sigma(z)\}_{\sigma \in I}$ represents the same cohomology class as $\{G_\sigma(z)\}_{\sigma \in I}$ defined above by $\mu_{\lambda,\sigma}$ the proof is completed. But this follows immediately from the following fact:

We can write $\mu_\lambda^\sigma = \sum_{\sigma' \in I} \mu_{\lambda,\sigma'}^\sigma$ such that each $\mu_{\lambda,\sigma'}^\sigma$ satisfies (2) and has its support in $\Gamma_{\sigma'}$ and that $\mu_{\lambda,\sigma'} = \sum_{\sigma \in I} \mu_{\lambda,\sigma'}^\sigma$. Then $\int_{V_\lambda} {}^t\partial_\lambda(\zeta, D_\zeta) \exp \langle \sqrt{-1} z, \zeta \rangle \cdot d\mu_{\lambda,\sigma'}^\sigma(\zeta)$ is holomorphic in the convex hull of W_σ and $W_{\sigma'}$. This implies that $\{G_\sigma(z)\}_{\sigma \in I}$ and $\{F_\sigma(z)\}_{\sigma \in I}$ are congruent modulo $dC^{n-1}(\mathcal{U}, \mathcal{U}', \mathcal{O}_P^{r_0})$ in $C^n(\mathcal{U}, \mathcal{U}', \mathcal{O}_P^{r_0})$.

References

[1] Ehrenpreis, L.: A fundamental principle for systems of linear differential equations with constant coefficients and some of its applications. Proc. Intern. Symp. on Linear Spaces, Jerusalem, pp. 161–174 (1961).

[2] ——: Fourier Analysis in Several Complex Variables. Wiley-Interscience, New York (1970).

[3] Kaneko, A.: Representation of hyperfunctions by measures and some of its applications. J. Fac. Sci. Univ. Tokyo Sect IA, **19**, 321–357 (1972).

[4] ——: Fundamental principle and extension of solution of linear differential equations with constant coefficients. Hyperfunctions and Pseudo-Differential Equations, Proceeding of a Conference at Katata, 1971, Lecture Notes in Math. No. 287, Springer, Berlin, pp. 122–134 (1973).

[5] Komatsu, H.: Resolution by hyperfunctions of sheaves of solutions of differential equations with constant coefficients. Math. Ann., **176**, 77–86 (1968).

[6] Palamodov, V. P.: Linear Differential Operators with Constant Coefficients (in Russian, Japanese translation, Yoshioka, Kyoto, 1973). Nauka Moscow (1967).