71. A Note on Nonlinear Differential Equation in a Banach Space

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(Comm. by Kinjirô KUNUGI, M. J. A., April 18, 1974)

1. Let E be a Banach space with the dual space E^*. The norms in E and E^* are denoted by $\| \|$. We denote by $S(u, r)$ the closed sphere of center u with radius r.

It is our object in this note to give a sufficient condition for the existence of the unique solution to the Cauchy problem of the form

(1.1) $$u'(t)=f(t, u(t)), \qquad u(0)=u_0 \in E,$$

where f is a E-valued mapping defined on $[0, T]\times S(u_0, r)$.

We compare the differential equation (1.1) with the scalar equation

(1.2) $$w'(t)=g(t, w(t)),$$

where $g(t, w)$ is a function defined on $(0, a]\times[0, b]$ which is measurable in t for fixed w, and continuous monotone nondecreasing in w for fixed t. We say w is a solution of (1.2) on an interval I contained in $[0, a]$ if w is absolutely continuous on I and if $w'(t)=g(t, w(t))$ for a.e. $t \in I^\circ$, where I° is the set of all interior points of I.

We assume that g satisfies the following conditions:

(i) There exists a function m defined on $(0, a)$ such that $g(t, w) \leqq m(t)$ for $(t, w) \in (0, a)\times[0, b]$ and for which m is Lebesgue integrable on (ε, a) for every $\varepsilon>0$.

(ii) For each $t_0 \in (0, a]$, $w\equiv 0$ is the only solution of the equation (1.2) on $[0, t_0]$ satisfying the conditions that $w(0)=(D^+w)(0)=0$, where D^+w denotes the right-sided derivative of w.

2. Let g be as in Section 1. Then we have the following lemmas.

Lemma 2.1. *Let $\{w_n\}$ be a sequence of functions from $[0, a]$ to $[0, b]$ converging pointwise on $[0, a]$ to a function w_0. Let $M>0$ such that $|w_n(t)-w_n(s)|\leqq M|t-s|$ for $s, t \in [0, a]$ and $n\geqq 1$. Suppose further that for each $n\geqq 1$*

$$w_n'(t)\leqq g(t, w_n(t)) \qquad \textit{for } t \in (0, a)$$

such that $w_n'(t)$ exists. Then w_0 is a solution of (1.2) *on* $[0, a]$.

For a proof see [4].

Lemma 2.2. *Let $M>0$ and let $\{w_n\}$ be a sequence of functions from $[0, a]$ to $[0, b]$ with the property that $|w_n(t)-w_n(s)|\leqq M|t-s|$ for all $s, t \in [0, a]$ and $n\geqq I$. Let $w=\sup_{n\geqq 1} w_n$, and suppose that $w_n'(t) \leqq g(t, w_n(t))$ for $t \in (0, a)$ such that $w_n'(t)$ exists. Then w is a solution of* (1.2) *on* $[0, a]$.

For a proof see [2].

Lemma 2.3. *Let w be an absolutely continuous function from $[0,a]$ to $[0,b]$ such that $w(0)=(D^+w)(0)=0$ and $w'(t)\leqq g(t,w(t))$ for $t\in(0,a)$ such that $w'(t)$ exists. Then $w\equiv 0$ on $[0,a]$.*

The proof of this lemma is quite similar to that of Theorem 2.2 in [1] and is omitted.

3. For each u in E let $F(u)$ denote the set of all x^* in E^* such that $(u,x^*)=\|u\|^2=\|x^*\|^2$, where (u,x^*) denotes the value of x^* at u.

Theorem. *Let f be a strongly continuous mapping of $[0,T]\times S(u_0,r)$ into E such that*

(3.1) $$2\,\mathrm{Re}\,(f(t,u)-f(t,v),x^*)\leqq g(t,\|u-v\|^2)$$

for (t,u), $(t,v)\in(0,T]\times S(u_0,r)$ and for some $x^*\in F(u-v)$, where g satisfies the conditions in Section 1 with $a=T$ and $b=\mathrm{Max}\,\{4r^2, 8rMT\}$. Then (1.1) has a unique strongly continuously differentiable solution u defined on some interval $[0,T_0]$.

Proof. Since f is strongly continuous on $[0,T]\times S(u_0,r)$ there exist constants $0<r_0\leqq r$, $0<T_1\leqq T$ and $M>0$ such that $\|f(t,u)\|\leqq M$ for $(t,u)\in[0,T_1]\times S(u_0,r_0)$. Let $T_0=\mathrm{Min}\,\{r_0/M, T_1\}$ and let n be a positive integer. We set $t_0^n=0$, and $u_n(t_0^n)=u_0$. Inductively, for each positive integer i, define $\delta_i^n, t_i^n, u_n(t_i^n)$ as follows:

(3.2) $$\delta_i^n\geqq 0,\qquad t_{i-1}^n+\delta_i^n\leqq T_0.$$

If

(3.3) $$\|v-u_n(t_{i-1}^n)\|\leqq M\delta_i^n \quad\text{and}\quad |t-t_{i-1}^n|\leqq\delta_i^n,$$

then $\|f(t,v)-f(t_{i-1}^n,u_n(t_{i-1}^n))\|\leqq 1/n$.

(3.4) $$\|u_n(t_{i-1}^n)-u_0\|\leqq r_0,$$

and δ_i^n is the largest number such that (3.2) to (3.4) hold. Define $t_i^n=t_{i-1}^n+\delta_i^n$ and define for each $t\in[t_{i-1}^n,t_i^n]$

(3.5) $$u_n(t)=u_n(t_{i-1}^n)+\int_{t_{i-1}^n}^t f(s,u_n(t_{i-1}^n))ds.$$

Then we have

(3.6) $$\|u_n(t)-u_n(s)\|\leqq M|t-s|,\ \|u_n(t)-u_0\|\leqq r_0 \qquad \text{for } s,t\in[0,T_0],$$

and $t_N^n=T_0$ for some positive integer $N=N(n)$. For some detail see [6] and [3].

Let $w_{mn}(t)=\|u_m(t)-u_n(t)\|^2$ for $m>n\geqq 1$ and $t\in[0,T_0]$. Obviously $w_{mn}(0)=0$, and $|w_{mn}(t)-w_{mn}(s)|\leqq 8r_0M|t-s|$ for $s,t\in[0,T_0]$. For each $t\in(0,T_0)$ there exist positive integers i and j such that $t\in(t_{j-1}^m,t_j^m)$ and $t\in(t_{i-1}^n,t_i^n)$. By Lemma 1.3 in [5] and (3.5) we have

(3.7) $$\begin{aligned} w'_{mn}(t)&=2\,\mathrm{Re}\,(u'_m(t)-u'_n(t),x^*_{mn}(t))\\ &=2\,\mathrm{Re}\,(f(t,u_m(t_{j-1}^m))-f(t,u_n(t_{i-1}^n)),x^*_{mn}(t))\\ &\leqq g(t,w_{mn}(t))+2(1/m+1/n)\,\|u_m(t)-u_n(t)\|\\ &\leqq g(t,w_{mn}(t))+8r_0/n \end{aligned}$$

for a.e. $t\in(0,T_0)$ and for some $x^*_{mn}(t)\in F(u_n(t)-u_n(t))$.

Let $w_n(t)=\sup_{m>n}w_{mn}(t)$ for $t\in[0,T_0]$. Then obviously $w_n(0)=0$

for $n \geqq 1$. By Lemma 2.2 and (3.7) we have

(3.8) $$|w_n(t)-w_n(s)| \leqq 8r_0M|t-s| \qquad \text{for } s, t \in [0, T_0],$$

and

(3.9) $$w_n'(t)=g(t, w_n(t))+8r_0/n \qquad \text{for a.e. } t \in (0, T_0).$$

On the other hand, $0 \leqq w_n(t) \leqq w_n(0) + 8r_0Mt \leqq 8r_0MT_0$ for $n \geqq 1$ and $t \in [0, T_0]$. Thus the sequence $\{w_n\}$ is equicontinuous and uniformly bounded, and hence it has a subsequence $\{w_{n_j}\}$ converging uniformly on $[0, T_0]$ to a function w, and obviously $w(0)=0$. It follows from (3.9) and Lemma 2.1 that $w'(t)=g(t, w(t))$ for a.e. $t \in (0, T_0)$.

We shall next show that $(D^+w)(0)=0$. Since f is continuous at $(0, u_0)$, given $\varepsilon>0$ we can find $\delta>0$ such that $\|f(t, u)-f(0, u_0)\|<\varepsilon$ whenever $0 \leqq t \leqq \delta$ and $\|u-u_0\| \leqq \delta$. Let $\delta_0=\text{Min}\{\delta, \delta/M\}$. Then, by (3.6), $\|u_n(t)-u_0\| \leqq \delta_0$ for all n and $t \in [0, \delta_0]$, and therefore $\|f(t, u_m(t)) - f(t, u_n(t))\|<2\varepsilon$ whenever $m>n \geqq 1$ and $t \in [0, \delta_0]$. By (3.3) and (3.7) we have

$$\begin{aligned} w_{mn}'(t) &= 2 \operatorname{Re}\,(f(t, u_m(t_{j-1}^m))-f(t, u_n(t_{i-1}^n)), x_{mn}^*(t)) \\ &\leqq 4r_0\,\|f(t, u_m(t_{j-1}^m))-f(t, u_n(t_{i-1}^n))\| \leqq 8r_0(\varepsilon+1/n) \\ &\qquad\qquad \text{for a.e. } t \in (0, \delta_0), \end{aligned}$$

and hence, by integrating the above inequality, we have

$$0 \leqq w_{mn}(t) \leqq 8r_0(\varepsilon+1/n)t,$$

whence $(D^+w)(0)=0$. From Lemma 2.3, we deduce now that $w \equiv 0$, and this implies that the sequence $\{u_n\}$ is uniformly convergent on $[0, T_0]$. The limit of this sequence satisfies

$$u(t)=u_0+\int_0^t f(s, u(s))ds \qquad \text{for } t \in [0, T_0]$$

(see [3]). Consequently u is a strongly continuously differentiable solution of (1.1) on $[0, T_0]$.

Let v be another strongly continuously differentiable solution of (1.1) on $[0, T_0]$. Let $z(t)=\|u(t)-v(t)\|^2$. Then obviously $z(0)=0$, and

$$z'(t)=2 \operatorname{Re}\,(f(t, u(t))-f(t, v(t)), x^*(t)) \leqq g(t, z(t))$$

for a.e. $t \in (0, T_0)$ and for some $x^*(t) \in F(u(t)-v(t))$. The fact $(D^+z)(0)=0$ follows from $0 \leqq z(t)/t = t\|(u(t)-v(t))/t\|^2 \to 0$ as $t \downarrow 0$. Therefore by Lemma 2.3 $z \equiv 0$, and the proof is complete.

References

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