68. *A Theorem on Riemannian Manifolds of Positive Curvature Operator*

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(Comm. by Kinjirô KUNUGI, M. J. A., April 18, 1974)

Let M^n $(n>2)$ be a compact orientable Riemannian manifold. If there exists a positive constant k such that

$$(*) \qquad -R_{hjil}u^{hj}u^{il} \geqq 2ku_{ij}u^{ij}$$

holds good for any skew symmetric tensor u_{ij} at any point, then M^n is called to be of positive curvature operator. M. Berger [1] has proved $b_2(M)=0$ for the second Betti number of such manifolds, and then $b_i(M)=0$ by D. Meyer [3] for $i=1,\cdots,n-1$.

The purpose of this note is to prove the following.

Theorem. *If a compact orientable Riemannian manifold M^n $(n>2)$ of positive curvature operator satisfies*

$$(\sharp) \qquad \nabla^h R_{hjil}=0,$$

then M^n is a space of constant curvature.

We remark that the condition $(\sharp)$ is satisfied when M^n has one of the following properties:

(i) the Ricci tensor is proportional to the metric tensor,

(ii) the Ricci tensor is parallel,

(iii) conformally flat, and the scalar curvature is constant.

Denoting the Ricci tensor by $R_{ji}=R_{hji}{}^h$ we define a scalar function K by

$$K=R_{lm}R^{ljih}R^m{}_{jih}+(1/2)R^{lmpq}R_{lmjh}R^{jh}{}_{pq}+2R^{ljmh}R_{lpmq}R^p{}_j{}^q{}_h.$$

Then we have

Lemma 1 ([2], [4]). *In a compact orientable Riemannian manifold, the integral formula*

$$\int_M \left\{K-|\nabla^h R_{hjil}|^2\right\}d\sigma=-\frac{1}{2}\int_M |\nabla_p R_{hjil}|^2\, d\sigma$$

holds good, where $|A_{jih}|^2=A_{hji}A^{hji}$, etc.

As it follows from $(\sharp)$ that

$$\int_M K d\sigma=-\frac{1}{2}\int_M |\nabla_p R_{hjil}|^2\, d\sigma \leqq 0,$$

we shall calculate K under the condition $(*)$.

Let P be any point of M^n and consider all quantities with respect to an orthonormal base field around P. For fixed k,j,i,h we define a local skew symmetric tensor field $u_{lm}^{(kjih)}$ by

$$u_{lm}^{(kjih)}=R_{ljih}\delta_{mk}+R_{klih}\delta_{mj}+R_{kjlh}\delta_{mi}+R_{kjil}\delta_{mh}-R_{mjih}\delta_{lk}-R_{kmih}\delta_{lj}-R_{kjmh}\delta_{li}-R_{kjim}\delta_{lh}.$$

Then, after long but simple calculations we can get

Lemma 2. $\sum_{\substack{k,j,i,h\\l,m,p,q}} R_{lmpq}u_{lm}^{(kjih)}u_{pq}^{(kjih)}=-16K.$

Lemma 3. $\sum_{\substack{k,j,i,h\\l,m}} u_{lm}^{(kjih)}u_{lm}^{(kjih)}=8(n-1)\,|W_{kjih}|^2,$

where W_{kjih} is the projective curvature tensor:

$$W_{kjih}=R_{kjih}+\frac{1}{n-1}(R_{ki}\delta_{jh}-R_{ji}\delta_{kh}).$$

Now, by virtue of Lemma 2, 3 and (*) we have

$$K\geqq(n-1)k\,|W_{kjih}|^2\geqq 0.$$

Hence it follows that

$$0\leqq(n-1)k\int_M |W_{kjih}|^2\,d\sigma\leqq-\frac{1}{2}\int_M |\nabla_p R_{kjih}|^2\,d\sigma\leqq 0.$$

Consequently we have $W_{kjih}=0$ and hence M^n is of constant curvature.

References

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