

66. Quantitative Properties of Analytic Varieties Complex Analytic De Rham Cohomology. II

By Nobuo SASAKURA
Tokyo Metropolitan University

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This note is a continuation of [3]. The purpose of this note is to outline our recent results on certain quantitative properties of *real* analytic varieties. Details will appear elsewhere. The results will provide a *topological* key to the complex analytic De Rham cohomology theory. In what follows we are basically concerned with asymptotic and division properties of C^∞ -differentiable differential forms with respect to given real analytic varieties. In this note we always mean by a variety a real analytic variety and we abbreviate the word C^∞ -differentiable as C^∞ . The symbols $L, N(Q, V)$, etc., have the same meanings as in [3]. For a fixed system of coordinates $(x) = (x_1, \dots, x_n)$ of R^n , $D_K = \partial^{|K|} / \partial x^K$, where $K = (k_1, \dots, k_n)$, $x^K = x_1^{k_1} \dots x_n^{k_n}$. Let \mathcal{D} be a domain in R^n and W a closed subset of \mathcal{D} . A C^∞ -function f in $\mathcal{D} - W$ is said to be of *polynomial growth with respect to W* if, for each K , there exists a couple a_K such that $|D_K f(Q)| \leq a_K \cdot d(Q, W)^{-a_K}$. A C^∞ -form $\varphi = \sum_J \varphi_J dx^J$ in $\mathcal{D} - W$ will be said to be of *polynomial growth with respect to W* if each coefficient φ_J is of polynomial growth.

Let (U, V, P) be a datum composed of a domain U in R^n , a variety V in U and a point P in V . This datum will be fixed throughout this note. First we state our results in terms of varieties in question and of coordinates (x) .

n.1. C^∞ -thickenings and their quantitative properties. Consider a proper subvariety $V' \ni P$ of V in addition to the datum (U, V, P) . For a couple σ , let $N_\sigma(V : V')$ denote the neighbourhood of $V - V'$ defined by $N_\sigma(V : V') = \bigcup_{Q \in V - V'} N_\sigma(Q : V')$. A neighbourhood N of $V - V'$ is called a C^∞ -*thickening* of $V - V'$, if $H^*(V - V' : \mathbf{R}) \cong H^*(N : \mathbf{R})$. Let $\{N_j : j \in \mathbf{Z}\}$ be a direct system of C^∞ -thickenings with respect to the inclusion relation satisfying the following conditions:

- (1) For any N_j there exists a couple σ_j such that $N_j \supset N_{\sigma_j}(V : V')$.
- (2) For an arbitrary σ , $N_j \subset N_\sigma(V : V')$ for a sufficiently large j .

For a neighbourhood N of $V - V'$, $\Omega(N)$ denotes the ring of C^∞ -differential forms in N . Moreover, we understand by $\Omega(N : V')$ the subring of $\Omega(N)$ composed of those forms which are of polynomial growth with respect to V' . Given a direct system $\{N_j : j \in \mathbf{Z}\}$ of C^∞ -

thickenings of $V - V'$, we let $\hat{\Omega}(V : V')$ be the direct limit : $\lim \cdot \text{dir} \cdot_j \Omega(N_j ; V')$. This ring $\hat{\Omega}$ is a differential ring in an obvious manner. Our first result is as follows.

Lemma 1. *For a fixed datum (U, V, V', P) , there exists a neighbourhood U' of $P : U \supset U'$ and a direct system $\{N_j : j \in \mathbf{Z}\}$ of C^∞ -thickenings of $(V - V') \cap U'$ such that*

$$(I) \quad H^*(U' \cap (V - V') : \mathbf{R}) \cong \mathcal{H}(\hat{\Omega}(V \cap U' : V' \cap U'))$$

holds.

n.2. Quantitative properties of retraction maps. We start with the datum (U, V, P) . Consider a subvariety $D' \ni P$ of U such that $D' \not\supset V$. Let I be the interval $[0, 1]$. A continuous map $\tau : I \times U \rightarrow U$ is a *retraction* of (U, V, D') to P if the following conditions are satisfied. (i) $\tau(1, Q) = Q$ for $Q \in U$, (ii) $\tau(0, Q) = P$ for $Q \in U$, (iii) $\tau : I \times V \subset V$ and $\tau : I \times D' \subset D'$. When we fix a datum (U, V, D', P) , we always assume that a map τ is C^∞ -differentiable in $(0, 1] \times (U - D')$. We say that a retraction map τ has *algebraic quantitative property with respect to (V, D')* if the following conditions are satisfied.

(II)₁ There exist triples of positive numbers $(\beta) = (\beta_1, \beta_2, \beta_3)$ and $(\beta') = (\beta'_1, \beta'_2, \beta'_3)$ such that the following *distance preserving property* with respect to V holds for each point $Q \in U$.

$$\beta_1 \cdot d(Q, V)^{\beta_2} \cdot \rho^{\beta_3} \leq d(Q_\rho, V) \leq \beta'_1 \cdot d(Q, V)^{\beta'_2} \cdot \rho^{\beta'_3},$$

where ρ is in $(0, 1]$ and $Q_\rho = \tau(\rho, Q)$.

(II)₂ For each pair (k, K) , $k \in \mathbf{Z}^+$, $K \in (\mathbf{Z}^+)^n$, there exists a triple $\gamma(k, K)$ such that the following inequality holds for each point $Q \in U - D'$:

$$|(\partial^k / \partial \rho^k) \cdot D_K x_j(Q_\rho)| \leq \gamma(k, K)_1 \cdot d(Q, D')^{-\tau(k, K)_2} \cdot \rho^{-\tau(k, K)_3}.$$

Now our second assertion is as follows.

Lemma 2 (Quantitative properties of retractions). *For a given datum (U, V, P) we find a neighbourhood U' of P and varieties D'_j ($j=1, \dots, m$) in U' such that*

(i) $\cap_j D'_j$ is a proper subvariety of V ,

and

(ii) for each D'_j there exists a retraction τ_j of $(U', V \cap U', D'_j)$ having algebraic quantitative property with respect to (V, D'_j) .

Remark 1. In both Lemmas 1 and 2 the set of neighbourhoods of P are cofinal with the set of neighbourhoods of P .

Remark 2. Let τ be a retraction of a pair (U, V) to P . Then τ is, in general, not C^∞ in the whole set $(0, 1] \times U$. In our Lemma 2, varieties D'_j , outside which τ_j is C^∞ , arise from two reasons: (i) The existence of the singular locus of V . (ii) The existence of singularities of maps which will be considered below.

Now we briefly indicate the relation between the above lemmas and our original problem of the complex analytic De Rham cohomology.

Lemma 1 is a C^∞ -analogue to the isomorphism: $(R^*i_*C)_P \cong \mathcal{H}(\hat{\Omega}(*D)_P)$ (cf. [3]) and is used in our proof of that isomorphism. Lemma 2 is used to show a *division property of integrations of differential forms* in the following manner: Start with the datum (U, V, P) . Assume that V is the zero locus of a *real analytic function* f . A C^s -differentiable form φ is said to be divisible by f m -times if each coefficient φ_j is written as $\varphi_j = \varphi'_j \cdot f^m$ with a C^s -function φ'_j . Roughly speaking, our problem is of the following type.

Find a domain $U' : U \supset U' \ni P$ and a $C^{s'}$ -form φ' in U' in such a way that

$$d\varphi' = \varphi, \quad \text{and} \quad \varphi' \quad \text{is divisible } m'\text{-times by } f.$$

A precise formulation of this problem will be given elsewhere. It is not difficult to see that Lemma 2, combined with the standard method of proving Poincaré lemma (cf. De Rham [2]), plays a key role in the above problem.

Lemmas 1 and 2 are of intrinsic nature to the given data (U, V, P) and (U, V, V', P) . In our discussions of these lemmas some other materials are introduced. Materials introduced will be explained below. Several interesting problems arise from the materials introduced. Here we shall explain briefly our methods employed in discussing our lemmas. (For details see [7].) We first associate with V a series $\mathcal{T} = (U^i, V^i, D^i)$ ($i=1, \dots, n$) of domains U^i in $R^i = \{(x_1, \dots, x_i)\}$ and varieties (V^i, D^i) in U^i . We assume the condition $(U^n, V^n, D^n) = (U, V, D)$. Also we assume that coordinates (x) are obtained from the original one by a suitable linear transformation. Moreover, a series $\mathcal{S} = \{S^i\}_i$, $S^i = \{S^i_j\}$ of stratifications of U^i is attached to \mathcal{T} . We impose certain *compatibility conditions* between \mathcal{T}, \mathcal{S} and the natural projections $\pi_{ii'} : (x^{i'}) \rightarrow (x^i)$. Furthermore, we associate with S^i_j finite sets of real analytic functions \mathcal{F}^i_j in U^i . A key point is that elements in \mathcal{F}^i_j *behave in a stable fashion along each stratum* S^i_j ; $S^i_{j'} \prec S^i_j$. An explicit expression of \mathcal{F}^i_j enables us to control quantitative behaviours of S^i_j along $S^i_{j'}$, $S^i_{j'} \prec S^i_j$ in connection with the map $\pi_{ii'}$. After introducing the above data $\mathcal{T}, \mathcal{S}, \mathcal{F}$ we consider corresponding facts to Lemmas 1, 2 in terms of the stratifications \mathcal{S}^i and investigate behaviours of C^∞ -thickenings, retractions, etc., under the maps $\pi_{ii'}$. Concerning Lemma 1 C^∞ -thickenings and their quantitative conditions are formulated in terms of stratifications \mathcal{S}^i . This procedure is regarded as '*localization*' procedure in our problem and has similarities to procedures in the well known *residue theory for smooth varieties*. Now recall that an open covering \mathcal{A} of a C^∞ -manifold is *simple* if each intersection $\bigcap_\lambda A_\lambda$, $A_\lambda \in \mathcal{A}$ is C^∞ -retractable (cf. A. Weil [5]). We associate with \mathcal{S}^i finite simple coverings \mathcal{A}^i . We impose certain quantitative conditions to \mathcal{A}^i . The introduction of the covering \mathcal{A}^i is our key point and leads easily to our Lemma 1. In our consideration

of Lemma 2 the following two points are particularly taken care of.

(i) The first point arises from our quantitative consideration: The standard methods for constructions of a retraction map τ of the pair (U, V) to P use the existence of certain vector fields (cf. S. Lojasiewicz [1], R. Thom [4], H. Whitney [6]). However, if we impose the quantitative conditions $(II)_{1,2}$ to the retraction τ , then arguments along the above basic methods cause a very subtle problem. The author does not know how to apply the above basic methods to our Lemma 2. The author's arguments are done along a line different from the above methods. Roughly we apply a method of the extension of C^∞ -functions to τ itself rather than to constructions of vector fields.

(ii) The second point arises from our introduction of varieties V_i and maps π_{ii} : If we do not consider any quantitative conditions to the retractions τ_i of (U^i, V^i, D^i) , then *lifting problem* of finding a retraction τ_{i+1} of $(U^{i+1}, V^{i+1}, D^{i+1})$ satisfying $\pi_{i,i+1} \cdot \tau_{i+1} = \tau_i \cdot \pi_{i,i+1}$ is not difficult. However, the conditions as $(II)_{1,2}$ imposed on τ_i, τ_{i+1} cause a delicate problem. In order to handle this problem we impose certain inequalities to $(\beta^i), (\beta'^i)$ and (γ^i) . Here $(\beta^i), \dots$ are triples with which inequalities $(II)_{1,2}$ are valid for the map τ .

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